Ferdinando A. Mussa-Ivaldi
Department of Brain and Cognitive Sciences

Neville Hogan
Department of Mechancial Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Integrable Solutions of Kinematic Redundancy via Impedance Control

Abstract

Problems arising when kinematically redundant manipulators are controlled using the Jacobian pseudoinverse are related to the nonintegrability of the standard pseudoinverse. This article presents a class of generalized inverses that have the property of being integrable within any simply connected, nonsingular region of the work space. Integrability is obtained by deriving the equations that describe an externally imposed motion, with the hypothesis that a compliance function is associated with each degree of freedom of the manipulator. The result is a weighted pseudoinverse containing a term that accounts for the nonlinear features of the forward kinematics. The relation of this integrable weighted pseudoinverse to the standard Moore-Penrose and other weighted pseudoinverses is discussed.

1. Introduction

It is widely recognized that a manipulator system with more controllable degrees of freedom than the minimum number required to describe spatial positioning tasks can offer significant advantages. For example, dextrous manipulation by robot hands or multiple coordinated arms appears to require a large number of degrees of freedom. Specifying the operation of these systems to perform a task with fewer degrees of freedom is an underdetermined, or "ill-posed," problem. Solutions based on the use of a generalized inverse of the manipulator's Jacobian have been proposed by several investigators (Whitney 1969; Liegeois 1977; Hollerbach and Suh 1985). However, Klein and Huang (1983) pointed out that this kind of control can drive the system to unpredictable configurations, a phenomenon that they related to the nonintegrability of a differential equation associated with the Jacobian pseudoinverse. In this article we show how this problem arises and present a new solution: a class of integrable weighted pseudoinverses for redundant manipulator systems. These solutions have the form of compliance-weighted pseudoinverses of the manipulator Jacobian. They are integrable within any simply connected region[1] of the work space in which the weighted pseudoinverses are not singular. We also show how these integrable solutions are related to more common generalized inverses such as the Moore-Penrose (MP) pseudoinverse.

2. Redundant Manipulators

The term *kinematic redundancy* is commonly used to indicate an excess of directly controlled degrees of freedom with respect to the number of parameters specified in a positioning task. Here, we will describe the task kinematics by an M-dimensional position vector $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M]^T$ that specifies the location of the end effector with respect to a system of environment-centered coordinates. The manipulator configuration is fully described by an N-dimensional array of generalized coordinates, $q = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N]^T$. Redundancy is expressed by the inequality $\mathbf{M} < \mathbf{N}$.

The forward kinematics is a vector map

$$x = x(q) \tag{1}$$

from configuration to end-effector position, which we will assume to be continuous and differentiable up to the second order in the entire work space. Because of the redundancy, this map cannot be uniquely inverted to obtain a configuration from an end-point position vector. Instead, given a position,

The International Journal of Robotics Research,
Vol. 10, No. 5, October 1991,

1. A simply connected region is a region within which every simple closed curve can be shrunk to a point by a continuous transformation without crossing the boundary of the region.

x_0, equation (1) defines an $\mathbf{N} - \mathbf{M}$-dimensional Riemannian manifold in the configuration space.

The differential transformation from configuration displacements to end-effector displacements[2] is:

$$dx = J(q)\, dq \qquad \text{with } J(q) = \frac{\partial x}{\partial q} \tag{2}$$

The matrix J is known as the Jacobian of the manipulator, and the above expression can be geometrically interpreted as the equation of an $\mathbf{N} - \mathbf{M}$-dimensional euclidean hyperplane that is tangent to the manifold of equation (1) at the point q. As is the case for the finite map, the differential expression (2) cannot be uniquely inverted for deriving a joint configuration change from a desired end-effector position change.

3. The Problem of Integrability

One approach to redundancy resolution is that of generating some global "inverse kinematic function" (Baker and Wampler 1987; Wampler 1987). This can be done, for example, by partitioning the configuration variables into a nonredundant, or "secondary," set and a complementary, or "primary," set. For the former set, a closed-form solution is assumed to be available, whereas the latter set is used as a parameter whose value can be related in any arbitrary way to the end-point locations. A different approach to redundancy resolution is based on the local inversion of the direct kinematics as it is expressed by the manipulator Jacobian, J. In this article, we focus on this latter approach, which involves the application of some generalized inverse, P, of the Jacobian matrix. Given a desired end-point displacement, dx, the corresponding configuration change, dq, is underdetermined. Provided that the Jacobian has full row rank, an expression that would yield a configuration change such that $dx = J(q)\, dq$ is given by:

$$dq = P_c(q)\, dx \tag{3}$$

with

$$P_c(q) = cJ^T(q)(J(q)cJ(q)^T)^{-1}$$

In this expression we have emphasized the fact that the P_c is, in general, a function of configuration, as it is obtained from operations involving the Jacobian matrix. The matrix c is a "weight matrix" that parameterizes the pseudoinverse: by using the weighted pseudoinverse P_c, together with solving an inverse kinematic problem, one minimizes the quadratic form $(dq^T c^{-1}\, dq)$. According to a more general algebraic definition (Ben-Israel and Greville 1980), a *generalized inverse*, G of a real-valued matrix A is a matrix that satisfies any one of the following conditions:

$$AGA = A \tag{4}$$

$$GAG = G \tag{5}$$

$$(AG)^T = AG \tag{6}$$

and

$$(GA)^T = GA. \tag{7}$$

There exists only one matrix that satisfies to all four conditions. This is the Moore-Penrose (MP) pseudoinverse of A. The weighted pseudoinverse, P_c, that we are considering here belongs to the broader class of generalized inverses which satisfy the first three conditions (4), (5) and (6). The MP pseudoinverse is obtained from P_c by setting $c = I$. The MP pseudoinverse, $P_I = J^T(J\,J^T)^{-1}$, minimizes the norm of the configuration change $(dq^T\, dq)$ compatible with the desired end-point displacement, dx.

A well-known problem with this approach (Klein and Huang 1983) arises when the end effector is moved along a closed path in the work space. Starting from a point x_s and with a configuration q_s and coming back to the same point x_s in one cycle, the manipulator may be in a different configuration $q_s + \Delta$, where the difference Δ is not necessarily zero. Unfortunately, Δ cannot be considered as an "error" associated with the use of finite steps in the computations; taking smaller steps does not necessarily reduce Δ. As cycles are repeated, these differences accumulate and may do so indefinitely, although sometimes a limiting configuration is reached. The disturbing impression associated with this situation is that of unpredictability; one is unable to establish the manipulator behavior at a more global scale than that at which equation (3) is defined.

These are symptoms of a very serious problem affecting the conventional pseudoinverse solutions: the problem of integrability. The vector expression (3) is a shorthand representation for a system of N equations known as total differential or Pfaffian equations (Levi-Civita 1926). Such a system is said to be integrable in a given domain if in this domain there exists a map,

$$q = q_c(x)$$

such that the expression (3) is the differential of this

2. We use the notation $(\partial f/\partial x)$ to indicate the Jacobian matrix of the vector function f, whose component i,j is $(\partial f_i/\partial x_j)$.

map—i.e., such that

$$P_c = \frac{\partial q_c}{\partial x}.$$

Conversely, if the system is not integrable, a unique global function from end-point position to configuration does not exist, and the configuration associated with a position of the end point will depend, among other things, on the path used to reach that position.

A necessary and sufficient condition for the integrability of the system in a simply connected domain is that:

$$\frac{dP_{c\backslash i,l}}{dx_k} = \frac{dP_{c\backslash i,k}}{dx_l} \tag{8}$$

within this domain. Here the total derivatives must be considered for taking into account not only the explicit dependency of P upon x, but also the implicit dependency, through $q_c(x)$. Unfortunately, this criterion is difficult to apply to our kinematic problem: the weighted pseudoinverse, P_c, depends on the configuration q whose relation with the end-effector position is unknown until the integrability problem has been solved. However, nonintegrability of standard weighted pseudoinverses (i.e., pseudoinverses with constant weighting matrices) has been demonstrated in particular cases (Klein and Huang 1983) and can be easily observed by iterating these pseudoinverses along closed paths.

As was pointed out by Baker and Wampler (1987), there is an equivalence between "cyclic" tracking algorithms and inverse kinematic functions. In the context of our discussion, this equivalence is merely the consequence of the fact that cyclic behavior within a simply connected domain is a necessary and sufficient condition for the integrability of a differential form.[3]

4. Derivation of the Integrable Weighted Pseudoinverses

In this article we define a new class of integrable weighted pseudoinverses without using criterion (8). Our method is based on two simple observations. The first (tautologic) observation is that if a vector map

$$y = f(x)$$

is differentiable in a given domain, then the differential map

$$dy = A\,dx \qquad \text{with } A = \frac{\partial f}{\partial x}$$

is integrable over the same domain. The second observation is that two integrable differential maps, $du = A_1\,dx$ and $dy = A_2\,du$, can be combined in a third map $dy = A_2A_1\,dx$, which is integrable provided that the range of the first map is contained within the domain of integrability of the second map.

Our specific goal is to set up an integrable map from the end-point displacement, dx, of a manipulator to a configuration displacement dq. The end-point displacement is an **M**-dimensional vector, whereas the configurational displacement is an **N**-dimensional vector, with $\mathbf{N} > \mathbf{M}$. Hence, by making use of the above observations, we start by defining a differentiable map, a compliance, from generalized forces to configurations. Then we combine this with two other integrable maps: one from end-point force-changes to generalized force-changes and the other from end-point displacements to end-point force-changes (the end-point stiffness). The final result is a single integrable map from end point to configuration displacements.

First, let us assume that for each degree of freedom, **i**, the generalized coordinate $\mathbf{q}_i$ is, at steady state, a differentiable function of the generalized forces—i.e., that

$$\mathbf{q}_i = \boldsymbol{\phi}_i(\mathbf{Q}_1, \ldots, \mathbf{Q}_N) \quad \text{and} \qquad \mathbf{c}_{ij} = \frac{\partial \boldsymbol{\phi}_i}{\partial \mathbf{Q}_j}; \qquad (-\infty < \mathbf{Q}_i < +\infty) \tag{9}$$

where $\mathbf{Q}_i$ is the components of the generalized force vector. We indicate by Q^N the **N**-dimensional space spanned by these components. It is evident that the **N** differential equations

$$\mathbf{dq}_i = \sum_{j=1}^{N} \mathbf{c}_{ij}\,\mathbf{dQ}_j \tag{10}$$

are, by construction, integrable and have (9) as particular solutions with a set of initial conditions $\mathbf{q}_i^0 = \boldsymbol{\phi}_i(\mathbf{Q}_1^0, \ldots, \mathbf{Q}_N^0)$. In vector/matrix notation the equation (10) becomes:

$$dq = c\,dQ \tag{11}$$

This expression relates changes of system configuration to changes in the (applied) generalized forces. Its physical meaning is that of a compliant behavior defined, at the level of each degree of freedom, either by the intrinsic mechanical properties of the

3. According to a fundamental theorem of differential calculus, the conditions expressed by eq. (8) are necessary and sufficient for the integrability of the differential form (3) in any simply connected domain within which P_c is nonsingular.

actuators or by the steady-state performance of their controllers. Because, by hypothesis, the functions (9) are differentiable in the entire Q^N, it follows that equation (11) is integrable in the entire Q^N. Note that the kinematic constraints of the system do not appear in any of the above equations. Thus the values that can be assumed by the generalized force vector in Q^N are unrestricted.

As we take into account the kinematic constraints arising from the geometric structure of a redundant manipulator, we obtain a restriction for the values that can be assumed by the generalized forces in Q^N. Given that the system kinematics is described by equation (2), the generalized force vector, Q, at a configuration q, is uniquely derived from the M-dimensional end-effector force, $F = [\mathbf{F}_1, \ldots, \mathbf{F}_M]^T$:

$$Q = J(q)^T F \tag{12}$$

Because the configuration, q, is itself a function of the generalized force [equation (9)], the above expression cannot be considered as an explicit definition of the map from end-effector force to generalized force. However, the equation

$$Q - J(q)^T F = 0$$

defines implicitly a differentiable map, $Q = Q(F)$, provided that

$$\det(I - \Gamma c) \neq 0. \tag{13}$$

Here, Γ is an $\mathbf{N} \times \mathbf{N}$ matrix containing the second derivatives of the transformation from configuration to end-effector coordinates. The element $\Gamma_{i,j}$ is:

$$\Gamma_{i,l} = \sum_{k=1}^{M} \frac{\partial^2 \mathbf{x}_k}{\partial \mathbf{q}_l \partial \mathbf{q}_i} \mathbf{F}_k. \tag{14}$$

With hypothesis (13) being satisfied, equation (12) defines an M-dimensional constraint surface, Σ, for the generalized forces, $\Sigma \subset Q^N$. Because the integrability of equation (11) applies to the whole Q^N, it also applies to Σ.[4] From a geometric standpoint, the problem of deriving a differential expression from equation (12) [i.e., an expression to be substituted for dQ in equation (11)] is equivalent to the problem of finding, for any value Q on Σ (i.e., for any generalized force satisfying the kinematic constraints), the plane passing through Q and tangent to Σ. In general, given a nonlinear transformation from generalized to end-effector coordinates, Σ is a curved Riemann surface whose tangent plane, Π_Q, at a point Q is obtained by differentiating equation (12). To this end, one must take into account that, given a change dF in the end-point force, the system will settle at a new steady-state configuration, $q + dq$. Accordingly, the Jacobian will change by an amount $dJ = J(q + dq) - J(q)$ and the change of generalized force, at steady state, is given by

$$dQ = J^T dF + dJ\, F. \tag{15}$$

To find a better expression for the last term on the right side, we expand the expression of its ith component ($\mathbf{i} = 1, \ldots \mathrm{N}$):

$$(dJ\, F)_i = \sum_{m=1}^{K} \left(\sum_{l=1}^{N} \frac{\partial^2 \mathbf{x}_m}{\partial \mathbf{q}_l \partial \mathbf{q}_i} \mathbf{dq}_l \right) \mathbf{F}_m = \sum_{l=1}^{N} \Gamma_{i,l}\, \mathbf{dq}_l.$$

In vector notation, the above expression becomes

$$dJ\, F = \Gamma\, dq.$$

By substituting equation (11) for dq, we finally obtain

$$dQ = (I - \Gamma c)^{-1} J^T dF, \tag{16}$$

which defines the tangent plane Π_Q to Σ. The above expression is integrable, as it has been obtained by differentiating equation (12). Note that with a linear geometry or with zero end-effector force, Π_Q coincides with Σ. Hence Γc can effectively be interpreted as a correction term that must be applied to the Jacobian to obtain the actual tangent planes with a curved (or Riemannian) constraint surface Σ.

Then by replacing the expression (16) for dQ in equation (11), we obtain the following expression for dq:

$$dq = c(I - \Gamma c)^{-1} J^T dF = (k - \Gamma)^{-1} J^T dF \tag{17}$$

with

$$k = c^{-1}.$$

This expression is also integrable, as it is the combination of two integrable maps ($dF \leftrightarrow dQ$ and $dQ \leftrightarrow dq$).

Finally, considering the transformation from generalized to end-point coordinates

$$dx = J(q)\, dq,$$

which is integrable by hypothesis, we obtain an integrable differential transformation

$$dx = C_e dF \quad \text{with} \quad C_e = J(k - \Gamma)^{-1} J^T, \tag{18}$$

which yields a map, $x = x(F)$, from end-point force to end-point position. Indicating by X the end-point

4. To show this, it is sufficient to consider any two points, Q_1 and Q_2, on Σ and to show that by integrating equation (11), the same result is obtained, regardless of the path chosen on Σ that joins these points. This must be true, because the two points and any path on Σ also belong to Q^N where integrability of equation (11) has been already proved.

work space, by Z the set of points in X in which $\det(C_e) = 0$, and by $X - Z$ the remaining portion of the work space, we have that the map (18) can be uniquely inverted in any simply connected region of $X - Z$. Accordingly, the differential transformation

$$dF = K_e\, dx \quad \text{with} \quad K_e = C_e^{-1} \tag{19}$$

is both well defined and integrable in the same region. By combining equations (17) and (19), we obtain a single expression for dq given dx:

$$dq = P_\phi\, dx \quad \text{with} \qquad P_\phi = (k - \Gamma)^{-1}J^T(J(k - \Gamma)^{-1}J^T)^{-1}. \tag{20}$$

The matrix P_ϕ is a weighted pseudoinverse of the Jacobian matrix [the weight matrix being $(k - \Gamma)^{-1}$]. The differential expression (20) is integrable by construction, as it has been obtained from a combination of integrable maps. We want to stress that integrability applies within simply connected regions without singularities (physically, regions in which an arbitrary displacement can be imposed on the end effector without encountering an infinite resistance). In contrast, conventional weighted pseudoinverses tend to be nonintegrable in any region of the work space, regardless of the presence or absence of singularities.

5. A Special Case: The Modified Moore-Penrose Solution

A special case of integrable weighted pseudoinverse is obtained when the compliance function is linear; i.e.,

$$q = cQ + q_0. \tag{21}$$

Then the integrable weighted pseudoinverse is a modified version of the standard weighted pseudoinverse, with constant weighting coefficients. In particular, if the compliance is the identity matrix, I, we have a modified Moore-Penrose (MMP) solution, which has the property of being integrable,

$$P_I = (I - \Gamma)^{-1}J^T(J(I - \Gamma)^{-1}J^T)^{-1}. \tag{22}$$

Note that Γ, like J, is a function of configuration. Because the MP pseudoinverse corresponds to the choice of minimizing the norm of the configuration displacements, the matrix Γ can be interpreted, in this context, as a correction for the curvature of the configuration space.

6. Passive versus Active Displacements

One physical interpretation of the integrable weighted pseudoinverse defined by equation (20) is that of a simulation of an externally imposed (or "passive") displacement of the manipulator. The steady-state behavior of the manipulator is defined by its compliance equation, and the task can be represented by an ideal position-servo operating on the end effector to bring it along the desired path. Then the degrees of freedom move in such a way as to minimize the potential energy stored in the compliance of the actuators. We have shown that the matrix Γ is essential to simulate this process correctly by taking into account the effects of the nonlinear geometries, which become significant as the end effector moves away from the equilibrium position. Neglecting Γ not only would generate an error, but also would, in general, result in a nonintegrable solution to the inverse kinematics.

The interpretation in terms of externally imposed (or "passive") displacements assumes a fixed equilibrium configuration for the manipulator. An alternative point of view, more directly related to manipulator control, is to consider Γ as an impedance component instead of a geometric term. We assume that steps are taken from equilibrium positions and that after each step the manipulator is brought to equilibrium. To achieve this, the N configuration variables, $\mathbf{q}_i$, must depend not only on the generalized forces, but also on N control inputs, u_i. Then the compliance function has the form:

$$q = \phi(Q, u). \tag{23}$$

The equilibrium configuration associated, at steady state, with an input u is

$$q_0(u) = \phi(0, u). \tag{24}$$

A change, du, of the input causes the equilibrium configuration to be updated by an amount:

$$dq_0 = \sigma\, du \qquad \text{with } \sigma = \frac{\partial\phi}{\partial u}. \tag{25}$$

Here σ is a local sensitivity matrix that we will assume to be nonsingular ($\det(\sigma) \neq 0$). Then, as the input changes smoothly in time, the above equation defines a sequence of static equilibria that has been termed a *virtual trajectory* (Hogan 1984).

The inverse kinematic problem becomes that of finding an appropriate sequence of inputs $u(t)$, given a desired trajectory of the end effector, $x_0(t)$, in the work space. One way to do this is to simulate passive displacements that will drive the joints away from equilibrium; then, at the end of each displacement, the input is modified to set equilibrium at the new manipulator configuration. Starting from an equilibrium position, x_0 [corresponding to $q_0(u)$], the weighted pseudoinverse approach corresponds to iterating a change of equilibrium, $dq_0 = P_\phi\, dx_0$, by updating the input with $du = \sigma^{-1}\, dq_0$. Because this process occurs about equilibrium, the matrix Γ is zero, and the method corresponds to applying the standard weighted pseudoinverse, $P = cJ^T(JcJ^T)^{-1}$. Because Γ is zero, there is no guarantee that the input update is integrable (i.e., that given a closed path of the hand, the input returns to its initial value).

Integrability can be recovered by specifying the manipulator compliance in such a way as to obtain the known integrable weighted pseudoinverse of equation (20). This corresponds to requiring that at each equilibrium configuration, the joint compliance, c, is:

$$c = (c'^{-1} - \Gamma)^{-1} \tag{26}$$

where c' is the compliance associated to some known differentiable function $\phi'(Q)$. Thus if the manipulator controller is able to modify the the compliance [equation (23)] functions, our method can be used to regularize the inverse kinematic problem through an appropriate choice of compliance parameters.

7. Remark on Singularities

In the discussion of kinematic inversion methods, it has become common practice to distinguish between two types of singularities: structural singularities and algorithmic singularities. The first is a property of hardware, the second of software. Structural singularities are those arising from the geometric properties (i.e., the "hardware") of a manipulator. They correspond to those work space locations at which the Jacobian becomes rank deficient. In contrast, algorithmic singularities result from the "extra" constraints set by the specific optimization method (i.e., the "software") that is used to regularize kinematic ill-posedness.

We believe that this distinction between structural and algorithmic singularities ceases to be significant in the context of our weighted pseudoinverse method. The weighted pseudoinverse operator, P_ϕ [eq. (20)], does indeed become singular whenever the end-point compliance, $C_\epsilon = (J(k - \Gamma)^{-1}J^T)^{-1}$, loses rank. This may happen only under two circumstances: (1) when the Jacobian loses rank, or (2) when the "linearized joint compliance," $c_{LIN} = (k - \Gamma)^{-1}$, loses rank.

One could erroneusly infer from equation (20) that a third case of singularity in P_ϕ may arise when $(k - \Gamma)$ loses rank. However, this is not the case, as it can be seen from the following argument. Let $\mathbf{D} = \det(k - \Gamma)$ and $A = \mathrm{Adj}(k - \Gamma)$. (Adj( ) denotes the adjoint matrix.) Then, it is

$$c_{LIN} = \mathbf{D}A,$$

and the expression for P_ϕ becomes

$$P_\phi = AJ^T(JAJ^T)^{-1}. \tag{27}$$

The above simplification, which results from the cancellation of $\mathbf{D}$ with $\mathbf{D}^{-1}$, is valid with an arbitrary small $\mathbf{D}$, as well as at the limit for $\mathbf{D} \to 0$. Therefore the points at which $\mathbf{D} = 0$ are not singularities for the weighted pseudoinverse P_ϕ.

Both singular cases, (1) and (2), correspond to a physical loss of mobility of the manipulator end point along some direction(s). The second condition, in which c_{LIN} becomes rank deficient, can be considered as an "algorithmic" singularity to the extent that the compliance matrix may reflect a specific impedance control strategy. However, a fundamental conjecture underlying impedance control is that it should be impossible through physical interaction with a machine to distinguish between the hardware and software contributions to its apparent behavior (Hogan 1985). In particular, this conjecture applies to our weighted pseudoinverse, which has been derived as the differential equation governing the passive displacement of a physical system.

We have succeeded in demonstrating that our weighted pseudoinverse is integrable within any simply connected domain of the work space. Of course, integrability ceases to hold along any end-point path enclosing one or more singular points (a well-known result of differential calculus). The converse is also true: if iterating P_ϕ along a closed path in the work space results in an open path in configuration space, then one or more singularities exist within that work space region. In principle this could provide a method for exploring the singularities in the work space. Note that this method would not work using the Moore-Penrose pseudoinverse or any other non-integrable operator.

8. Parameterized Postural Maps

The subscript ϕ in equation (20) indicates that our weighted pseudoinverse is parameterized by the choice of the compliance function. Because the differential map (20) is integrable by construction within simply connected regions of the work space, the compliance function parameterizes not only a local relation, but also a global map of postures within any such region:

$$q = q_\phi(x). \tag{28}$$

We want to stress that the domain over which the above global map can be derived by integrating the weighted pseudoinverse (20) is analogous to the "feasible work space" (Wampler 1987) over which inverse kinematic functions may be defined. Here the role of a compliance function, with respect to the redundant inverse kinematic problem, is that of selecting a unique map from end effector to configuration. Different kinematic solutions can be obtained by choosing different compliance functions. In particular, a way to generate different maps is by choosing different initial conditions for the differential equation (20). In the paradigm of passive motion described in the previous section, this corresponds to the choice of an equilibrium configuration q_0. Another possibility is to select different matrices of joint compliance, c. Both alternatives are special cases of compliance function selection.

Global postural maps can be generated in the form of look-up tables indexed by the end-point location. The simplest procedure to generate such a look-up table, corresponding to a specific compliance function, ϕ, is the following:

1. First, define the equilibrium configuration, $q_0 = \phi(0)$, and the corresponding end-point position, $x_0 = x(q_0)$. At this location, $F = 0$ and, consequently, $\Gamma = 0$.
2. Then, the other tabulated positions are obtained from a finite set of end-effector displacements, $\{dx_i\}$: $x_{n+1} = x_n + dx_n$. The corresponding configuration is derived by applying the weighted pseudoinverse; i.e., $q_{n+1} = q_n + P_{\phi,n}\, dq_n$. In order to derive the weighted pseudoinverse, $P_{\phi,n}$, one must know the value of the end-point force, F, at x_n [see equation (14)]. This is given by iterating $F_{n+1} = F_n + dF_n$ with $dF_n = K_e\, dx_n$. The term K_e in the last expression is given by equations (18) and (19). The initial condition is $F_0 = 0$.

9. A Numeric Example

Here we present the results of a numeric simulation of the integrable weighted pseudoinverse in the special case of the MMP [equation (22)]. We applied the algorithm to the same three-joint planar kinematic mechanism considered by Klein and Huang (1983). The joints are revolute, and their axes of rotations define the $\mathbf{z}$-axis of a cartesian coordinate system. The end point of the mechanism lies in the $\mathbf{x}$-$\mathbf{y}$ plane of this coordinate system, and the kinematics are defined by the transformation:

$$\begin{aligned} \mathbf{x} &= \mathbf{l}_1 \cos(\mathbf{q}_1) + \mathbf{l}_2 \cos(\mathbf{q}_1 + \mathbf{q}_2) \\ &\quad + \mathbf{l}_3 \cos(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\ \mathbf{y} &= \mathbf{l}_1 \sin(\mathbf{q}_1) + \mathbf{l}_2 \sin(\mathbf{q}_1 + \mathbf{q}_2) \\ &\quad + \mathbf{l}_3 \sin(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3). \end{aligned} \tag{29}$$

Here, $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$ are the lengths of the three arm segments ("humerus," "forearm," and "hand," respectively) and $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are the relative joint angles (at the "shoulder," "elbow," and "wrist," respectively).

Starting from an initial configuration the algorithm requires the tip to move along a predefined closed path. The motion is divided into steps: at each step of the tip the corresponding joint displacement is derived using either the MP or the MMP pseudoinverse. The desired step of the tip is divided by two until all the joint angle changes are less than a given threshold, δ. Thus the parameter δ defines the actual number of steps, NA, given a desired maximum angular change of each joint. At the end of the cycle we derive the tip position error, TPE, and the joint configuration error, JCE. The former is the cartesian distance between end and start location of the tip, $\text{TPE} = [(\mathbf{x}_{end} - \mathbf{x}_{start})^2 + (\mathbf{y}_{end} - \mathbf{y}_{start})^2]^{1/2}$. Similarly, JCE is the distance between final and initial configuration in joint angle coordinates. The MMP solutions were derived by assuming that the starting configuration was at equilibrium ($F = 0$) and by applying the algorithm described at the end of the previous section.

The results of some simulations with the MP and with the MMP are shown in Table 1. These data refer to a single trajectory of the tip. The starting configuration is $q_{start} = [45, 110, 0]^{T°}$. The corresponding starting location of the tip is $r = [-24.10, 42.34]^T$ cm (the manipulator link lengths are $\mathbf{l}_1 = 30$ cm, $\mathbf{l}_2 = 30$ cm, and $\mathbf{l}_3 = 20$ cm). The closed end-point path is a square with 20 cm sides (about 2% of the work space area).

The MP and the MMP are equally efficient in the kinematic inversions. This is shown by the fact that the TPEs generated by the two methods are approximately equal, and they decrease in the same way as the maximum configuration change δ decreases. The pattern of the JCEs highlights the difference between the standard MP and the MMP: as expected, the joint configuration error associated with the MP is significantly high (of the order of a few degrees) and is not changed by reducing δ. In contrast, the joint configuration error associated with the MMP decreases monotonically with δ, having the same order of magnitude.

Furthermore, the two methods yield different

Table 1. Iteration Over a Closed Path of the Moore-Penrose (MP) and of the Modified Moore-Penrose (MMP) Pseudoinverses*

MP
Starting configuration: (45, 110, 0) degrees
Starting tip position: (−24.10, 42.34) cm
Path length: 80 cm

δ (degrees)	N	TPE (cm)	JCE (degrees)
10^{-1}	1,940	4.91 10^{-2}	4.38
10^{-2}	18,202	5.23 10^{-3}	4.43
10^{-3}	192,175	5.02 10^{-4}	4.44
10^{-4}	1,965,093	4.85 10^{-5}	4.44

MMP
Starting configuration: (45, 110, 0) degrees
Starting tip position: (−24.10, 42.34) cm
Path length: 80 cm

δ (degrees)	N	TPE (cm)	JCE (degrees)
10^{-1}	1,964	4.76 10^{-2}	9.59 10^{-2}
10^{-2}	18,920	4.93 10^{-3}	1.00 10^{-2}
10^{-3}	192,730	4.79 10^{-4}	9.86 10^{-4}
10^{-4}	1,956,155	4.73 10^{-5}	9.61 10^{-5}

*The starting point and the path are identical in the two cases. The path is a square with the starting position at the lower right corner and is traversed in the counterclockwise direction. δ: maximum angle change allowed in a single step: N: number of iterations; TPE: tip position error, defined as $(\Delta X^2 + \Delta Y^2)^{1/2}$; JCE: joint configuration error, defined as $(\Delta q_1^2 + \Delta q_2^2 + \Delta q_3^2)^{1/2}$.

Table 2. Iteration of MP and MMP Over Closed Paths With Increasing Length*

MP
Starting configuration: (−30, 130, 60) degrees
Starting tip position: (1.98, 21.39) cm
$\delta = 10^{-1}$ degrees

Path length (cm)	N	TPE (cm)	JCE (degrees)
40	930	2.15 10^{-2}	4.79
80	1,807	3.90 10^{-2}	12.2
120	2,486	5.08 10^{-2}	18.5
160	3,226	8.63 10^{-2}	24.6

MMP
Starting configuration: (−30, 130, 60) degrees
Starting tip position: (1.98, 21.39) cm
$\delta = 10^{-1}$ degrees

Path length (cm)	N	TPE (cm)	JCE (degrees)
40	933	2.28 10^{-2}	5.68 10^{-2}
80	1,713	4.62 10^{-2}	1.09 10^{-1}
120	2,451	6.68 10^{-2}	1.47 10^{-1}
160	3,246	9.28 10^{-2}	2.27 10^{-1}

* The paths are squares with the starting position at the lower right corner. The maximum joint angle change, δ, is $10^{-1\circ}$ in all cases. See Table 1 for notation.

results when the path length increases at constant δ (Table 2). With both methods the tip position error and the joint configuration error increase linearly with the path length. However, with the MP, the slope of JCE vs. TPE is almost 300°/cm (regression coefficient, **r** = 0.964), whereas with the MMP, the same slope is two orders of magnitude less (**r** = 0.994). With the MP method the joint configuration error, after a single closed path of 160 cm, can get as big as 20°, whereas with the MMP method it remains limited to a few tenths of a degree at δ = 0.1°.

Taken together, these results indicate that the joint configuration error seen in the simulation of the MMP method is due only to the discretization and can be made arbitrarily small with the appropriate choice of δ. Furthermore, for any δ and for any path length, the effective number of steps, NA, carried out with each method is approximately the same. Hence in spite of the fact that a single step with the MMP method requires more operations than a step with MP, the order of magnitude of their computational costs is the same.

The possibility of generating different postural maps by selecting different compliance functions is demonstrated in Figures 1 and 2. Here we used a linear compliance function:

$$q = cT + q_0,$$

with a diagonal joint compliance matrix, c. The three diagonal terms of c specify, respectively, the joint stiffness at the shoulder, $c_{1,1}$, at the elbow, $c_{2,2}$, and at the wrist, $c_{3,3}$. Different postures corresponding to the same end-point locations can be

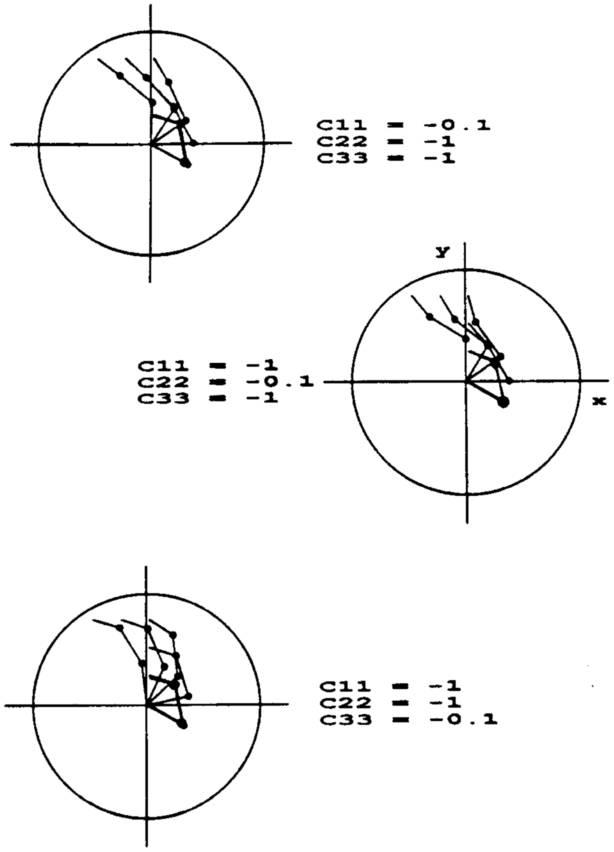

Fig. 1. Different postural maps generated by different choices of joint compliance (c11: shoulder; c22: elbow; c33: wrist). Darker lines indicate the equilibrium configuration.

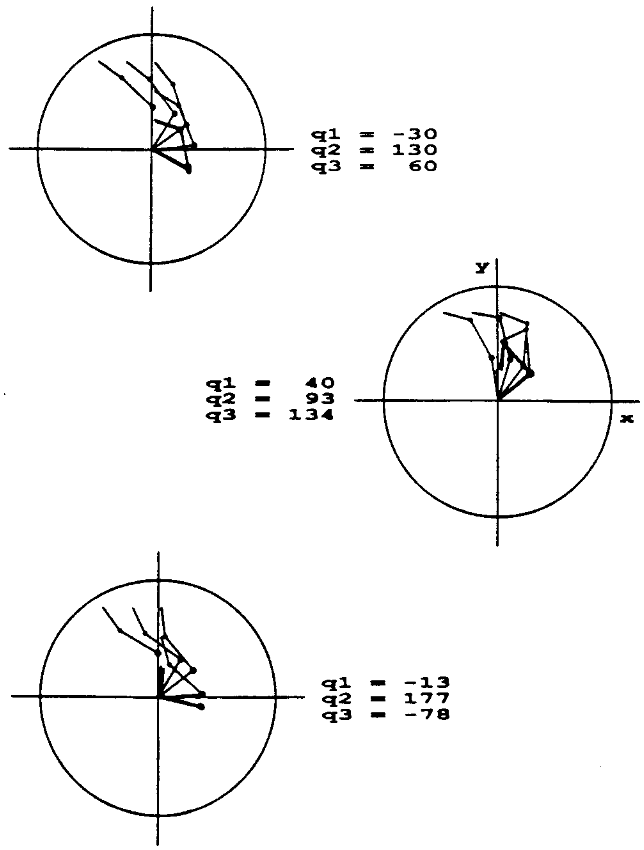

*Fig. 2. Different postural maps generated by different choices of initial configuration (*darker lines*). The three angles* q_1, q_2, *and* q_3 *are given in degrees and are relative angles;* q_1 = *upper arm to base,* q_2 = *forearm to upper arm,* q_3 = *hand to forearm.*

obtained by assigning different values to these coefficients (see Fig. 1). As can be intuitively understood, assigning a high compliance value to a joint tends to result in larger motions of that joint across the work space.

An alternative way of parameterizing postures is by selecting different initial configurations at a given end-point location (see Fig. 2). This can be done by null-space motions[5] about an initial configuration q_0.

In both cases, the differences among postural maps tend to diminish near the boundary of the work space. This is a direct consequence of the decrease in manipulability (Yoshikawa 1985)—i.e., of the loss of effective redundancy—in these regions.

5. Given a weighted pseudoinverse, P, of the Jacobian J, a null-space motion Δq_N can be obtained from $\Delta q_N = (I - PJ)\xi$, where ε is an arbitrary N-dimensional vector.

10. Conclusions

Many conventional solutions to the inverse kinematics problem for redundant manipulators involve the use of weighted generalized inverses of the Jacobian matrix. However, it has been recognized (Klein and Huang 1983) that weighted pseudoinverses with constant weighting matrices are not integrable. Consequently when one iterates one of these kinematic solutions, such as the Moore-Penrose pseudoinverse, the configuration corresponding to a desired end-effector location will, in general, depend on the path used to reach that location.

Other investigators have addressed this problem by combining null-space motions with the Moore-Penrose solution (Baillieul et al. 1984; Baillieul 1985). Here we take a different approach and demonstrate that there exists a subset of the weighted pseudoinverses whose elements are integrable. The weighting matrices of these integrable pseudoinverses are position dependent instead of constant. In this article, the issue of integrability has been cast in terms of manipulator mechanics: given a redundant mechanism, a kinematic task can be represented as an externally imposed displacement of the end effector. Then, assuming that a compliance function is associated to each degree of freedom, this externally imposed motion results into a unique and integrable motion in configuration space. In particular, if the joint compliance is mechanically conservative (i.e., if the local compliance matrix is symmetric), the joint configuration corresponding to the externally imposed position of the end effector is at minimum potential energy.

We have shown that the correct simulation of these externally imposed displacements requires considering the position dependence of the Jacobian matrix that characterizes nonlinear manipulator geometries. In this context, the use of conventional weighted pseudoinverses, with constant weighting matrices, would be invalid and misleading: invalid because changes of Jacobian must be taken into account in the differential transformations from end-effector forces to generalized forces, and misleading because conventional weighted pseudoinverses are sufficient to satisfy the kinematic constraints and therefore to generate configurational changes that "look like" actual solutions. Fortunately, kinematic nonlinearities are completely captured by a single correction matrix that has the physical meaning of an apparent joint stiffness term. This correction matrix vanishes at static equilibrium but becomes increasingly significant away from equilibrium, as the end-effector forces (induced by the joint compliance function) become larger. Our analysis indicates that the correction matrix is also sufficient to ensure the integrability of the weighted pseudoinverse.

This result can also be applied to the generation of active movements when these are obtained by modifying the torque/angle relation associated with each joint. For example, an active change of configuration is produced by varying the equilibrium position of each degree of freedom. Such a scheme can be implemented in a way that is consistent with an externally imposed displacement of the end effector (Mussa-Ivaldi et al. 1988): each joint equilibrium is actively changed by the same amount that would have been induced by the imposed motion of the end effector. Then integrability is ensured by an appropriate choice of the joint compliance throughout the work space (i.e., by including the correction matrix as an effective component of the joint impedance instead of as a compensation for nonlinear kinematics). In this context, the planning of inverse kinematics becomes a subset of impedance control (Hogan 1980, 1985): given a single task, different kinematic patterns can be generated by selecting different joint compliance functions.

Acknowledgments

This research was supported by the National Science Foundation, grant EET-8613104; by the Office of Naval Research, grant N00014/88/K/0372; by the National Institute of Neurological Disease and Stroke Research, grant NS09343; and by the Alfred P. Sloan foundation. We wish to thank Prof. Michael Jordan for helpful discussions and comments about this work.

References

Baillieul, J., Hollerbach, J., and Brockett, R. W. 1984. (Las Vegas). Programming and control of kinematically redundant manipulators. *Proc. 23rd IEEE Conf. on Decision and Control.* New York: IEEE, pp. 768–774.

Baillieul, J. 1985 (St. Louis). Kinematic programming alternatives for redundant manipulators. *Proc. IEEE Int. Conf. on Robotics and Automation.* New York: IEEE, pp. 722–728.

Baker, D. R., and Wampler, C. W. 1987 (Raleigh, N.C.). Some facts concerning the inverse kinematics of redundant manipulators. *IEEE Int. Conf. on Robotics and Automation.* New York: IEEE, pp. 604–609.

Ben-Israel, A., and Greville, T. N. E. 1980. *Generalized Inverses: Theory and Applications.* New York: R. E. Krieger Publishing Co.

Hogan, N. 1980 (San Francisco). Mechanical impedance control in assistive devices and manipulators. Paper TA-10-B. *Proc. 1980 Joint Automatic Control Conf.* New York: IEEE.

Hogan, N. 1984. An organising principle for a class of voluntary movements. *J. Neurosci.* 4(11):2745–2754.

Hogan, N. 1985. Impedance control: An approach to manipulation. Parts I–III. *ASME J. Dyn. Sys. Meas. Control.* 107(1):1–24.

Hollerbach, J. M., and Suh, K. C. 1985 (St. Louis). Redundancy resolution of manipulators through torque optimization. *Proc. IEEE Int. Conf. on Robotics and Automation.* New York: IEEE, pp. 1016–1021.

Klein, C. A., and Huang, C. 1983. Review of pseudoinverse control for use with kinematically redundant manipulators. *IEEE Trans. Sys. Man Cybernet.* SMC-13(2):245–250.

Levi-Civita, T. 1926. *The Absolute Differential Calculus. (Calculus of Tensors).* New York: Dover.

Liegeois, A. 1977. Automatic supervisory control of the configuration and behavior of multibody mechanisms. *IEEE Trans. Sys. Man Cybernet.* SMC-7(12):868–871.

Mussa-Ivaldi, F. A., Morasso, P., and Zaccaria, R. 1988.

Kinematic networks. A distributed model for representing and regularizing motor redundancy. *Biol. Cybernet.* 60(1):1–16.

Wampler, C. W. 1987 (Raleigh, N.C.). Inverse kinematics functions for redundant manipulators. *Proc. IEEE Int. Conf. on Robotics and Automation*. New York: IEEE, pp. 610–617.

Whitney, D. E. 1969. Resolved motion rate control of manipulators and human prostheses. *IEEE Trans. Man-Machine Syst.* MMS-10(2):47–53.

Yoshikawa, T. 1985 (St. Louis). Manipulability and redundancy control of robotic mechanisms. *Proc. IEEE Int. Conf. on Robotics and Automation*. New York: IEEE, pp. 1004–1009.