Biol. Cybern. 67, 479–489 (1992)

Biological Cybernetics



# From basis functions to basis fields: vector field approximation from sparse data

**Ferdinando A. Mussa-Ivaldi**

Department of Brain and Cognitive Sciences, Massachussetts Institute of Technology, Cambridge, MA 02139, USA

Received June 1, 1992/Accepted in revised form June 1, 1992

**Abstract.** Recent investigations (Poggio and Girosi 1990b) have pointed out the equivalence between a wide class of learning problems and the reconstruction of a real-valued function from a sparse set of data. However, in order to process sensory information and to generate purposeful actions living organisms must deal not only with real-valued functions but also with vector-valued mappings. Examples of such vector-valued mappings range from the optical flow fields associated with visual motion to the fields of mechanical forces produced by neuromuscular activation. In this paper, I discuss the issue of vector-field processing from a broad computational perspective. A variety of vector patterns can be efficiently represented by a combination of linearly independent vector fields that I call "basis fields". Basis fields offer in some cases a better alternative to treating each component of a vector as an independent scalar entity. In spite of its apparent simplicity, such a component-based representation is bound to change with any change of coordinates. In contrast, vector-valued primitives such as basis fields generate vector field representations that are invariant under coordinate transformations.

## 1 Introduction

From a mathematical perspective, learning a task is equivalent to establishing an associative map between input and output patterns from a collection of elements (or examples) of this mapping. Often, the learning process has been considered starting from tabula rasa – that is from a completely unstructured network of computational elements, each element implementing some elementary form of input-output relation (Hopfield 1982; Rumelhart et al. 1986). However, by focusing exclusively on such elementary computational modules as the McCulloch and Pitts neurons, one may miss one of the most relevant features of any learning system, that is the ability to take advantage of past experience as a building ground for developing new skills. Past experience may lead to the development of functional units which implement relatively complex input-output maps and which should be considered in their own right as computational primitives.

The idea of learning associative mappings by combining complex computational primitives has been developed into a formal theory by Poggio and Girosi (1990a, b). Their approach is based upon the equivalence between regularizing an ill-posed problem (Tikhonov and Arsenin 1977; Poggio et al. 1985) and reconstructing (or approximating) a surface by combining local basis funttions (Micchelli 1986; Powell 1987). From this theory, Poggio and Girosi derived a class of three-layer networks whose hidden units implement a set of basis functions.

Here, I propose to extend the same paradigm to the representation of vectorial maps. This is important in a wide range of computational tasks involving the reconstruction of a continuous vector field from a sparse set of examples. Usually, in the processing of vector-field information the data are available as physical coordinates, measured with respect to some reference system. In contrast, the most relevant information about a vector field may often be stated in terms of coordinate-invariant expressions involving relations among vector components, such as curl and divergence. In this case, the goal of vector field reconstruction is to derive these relations independently of the particular system of coordinates in which the data have been measured and encoded.

Poggio (1990) suggested that the paradigm of surface approximation by combining local basis functions may be relevant to biological information processing. However, in order to elaborate sensory information and to generate purposeful actions living organisms must deal not only with real-valued functions but also with vector-valued mappings. A well-known example of vector mapping in the domain of natural and artificial vision, is the "optical flow" (Gibson 1950; Horn 1986), that is the field of velocity vectors associated with the motion of a scene relative to an observer. Similarly, in the study of motor control, there is now evidence that

neural activities code and control a field of elastic forces generated by muscle viscoelastic properties (Hogan 1985; Mussa-Ivaldi et al. 1985; Bizzi et al. 1991; Shadmehr et al. 1992). Given the relevance of vector-field processing to diverse computational contexts, here I discuss this issue from the broad perspective established by the problem of approximating a spatial pattern of vectors. A particular application to motor control of the concepts presented in this paper can be found in Mussa-Ivaldi and Giszter (1992).

In the next sections I show that a variety of vector patterns is efficiently represented by a combination of linearly independent vector fields. I call these fields "basis fields" because they are directly related to the scalar basis functions used for reconstructing scalar maps from numerical examples. Basis fields offer an alternative to the direct approach where each component of a vector is seen as a scalar entity that is represented by a combination of scalar basis functions. This alternative is particularly useful when one is concerned with different coordinate representation of the same vector field. In fact, in spite of their apparent simplicity, component-based representations are bound to change with any change of coordinates. In contrast, by using vector-values primitives such as basis fields one may derive vector field representations that are invariant under coordinate transformations.

## 2 The vector-field approximation problem

Generally speaking, a vector field on a subset $\mathcal{E}$ of $\Re^N$ is a map that assigns to each point $x \in \mathcal{E}$ an element of the tangent space[1] of $\Re^N$ at $x$ (Spivak 1965). Here, I will describe a vector field on $\Re^N$ as a collection, $F(x)$, of $N$ real-valued maps, $F(x) = (F_1(x), F_2(x), \ldots, F_N(x))$ with the implicit but important assumption that such a collection belongs to a vector space: given two fields[2], $F^1(x)$ and $F^2(x)$ and a real number, $a$, the operation of addition and scalar multiplication are defined, so that:

$$F^1(x) + F^2(x) = (F_1^1(x) + F_1^2(x), \ldots, F_N^1(x) + F_N^2(x))$$

$$aF(x) = (aF_1(x), \ldots, aF_N(x)) .$$

There are important instances in which an array of real-valued maps cannot be described as a vector field. For example, a set of graphical features defined over an image may be expressed as an array of real-valued maps (e.g. contrast, brightness, etc.). For such an array, the addition and scalar multiplication as defined above may not be meaningful. Therefore, the array does not belong to a vector space although someone might still call it (improperly) a "feature vector". Similar arguments apply to the array of joint configuration variables. In a non-redundant manipulator, each joint angle is a scalar inverse-kinematics map over the space of end-point locations. However, vector addition is not properly defined for the array of joint angles because finite rotations in general do not add commutatively. Therefore the array of joint configurations does not have the structure of a vector field.

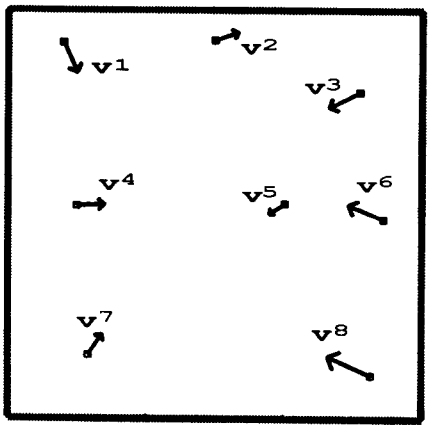


**Fig. 1.** Vector field approximation. 8 vectors (the "data") are presented at 8 distinct points. The square boundary delimits the approximation domain, $\mathcal{E}$

This paper is only concerned with the representation and approximation of continuous and differentiable vector fields[3]. I will address the issue of reconstructing and representing a vector field as an approximation problem. More precisely, let $\mathcal{E}$ be a region in $\Re^N$. Suppose that $M$ vectors, $v^1, v^2, \ldots, v^M$, are given at $M$ distinct points in $\mathcal{E}$, $x^1, x^2, \ldots, x^M$ (Fig. 1). Each point, $x^i$, is a collection of $N$ real numbers defined with respect to some coordinate system, that is

$$x^i = (x_1^i, x_2^i, \ldots, x_N^i) .$$

Similarly, each vector, $v^i$, is a collection of $N$ real-valued components, $(v_1^i, v_2^i, \ldots, v_N^i)$.

We may regard these vectors as samples of an unknown field, $\hat{F}(x)$, that is a vector-valued map

$$\hat{F} = \{v | v_i = \hat{F}_i(x); \quad i = 1, \ldots, N; \quad v_i \in \Re; \quad x \in \mathcal{E}\} .$$

Let, $F(x|c_1, c_2, \ldots, c_K)$, be a known vector-valued function of $x$ and of $K$ real-valued parameters, $c_i$. The goal of vector-field approximation is formally identical to the goal of the ordinary approximation of scalar maps (Rice 1964): given an appropriate definition of distance between two fields, find a set of parameter values which minimizes the distance between $\hat{F}(x)$ and $F(x|c_1, \ldots, c_K)$.

[1] Given a point $x$ of a manifold $M$, the tangent space of $M$ at $x$ is the vector space of all tangents to $M$ at $x$. The set, $TM$, of tangent spaces to all points of $M$ is called the *tangent bundle* of $M$. In this context, a vector field is a map $M \mapsto TM$

[2] Here, I adopt the convention of using superscript indices to indicate distinct multidimensional objects (points, vectors and matrices). In contrast, I will spell the components of a single multidimensional object by subscript indices

[3] The central ideas and results can be applied without much of a change to the broader domain of tensor fields

In the case of a scalar map, an approximating function may be represented as a weighted sum of pre-defined basis functions, $g_i(x)$, i.e.:

$$f(x|c_1, c_2, \ldots, c_K) = \sum_{i=1}^{K} c_i g_i(x) .$$

Poggio and Girosi (1990a,b) have shown that a set of basis functions can be determined from what is known a-priori about the smoothness properties of the unknown mapping. With a set of real-valued data, $y_1, \ldots, y_M$, defined at a set of points, $x^1, \ldots, x^M$ in $\Re^N$, scalar approximation involves finding a function, $f(x)$: $\Re^N \mapsto \Re$, which minimizes a functional

$$H[f] = \sum_{i=1}^{m} (y_i - f(x^i))^2 + \lambda \|Pf\|^2 \qquad (1)$$

where, $P$ is a linear differential operator and $\|\cdot\|$ is the $L_2$ norm. The first term on the right side of the above expression represents the "interpolation goal": the function $f$ should pass as close as possible to the data. The second term represents an addtional "regularization goal": $f$ should also minimize a cost functional, $\|Pf\|$, which represents what is known a-priori about the unknown mapping. The regularization parameter, $\lambda$, establishes a trade off between these two goals. Poggio and Girosi showed that a general solution to the approximation problem can be expressed as a sum of Green's functions, $G(x, x^i)$ of the self-adjoint operator $\hat{P}P$ ($\hat{P}$ is the adjoint of $P$) centered at the data points $x^i$:

$$f(x) = \sum_{i=1}^{m} c_i G(x, x^i) .$$

These Green's functions form a basis for an $m$-dimensional subspace of the space of smooth functions.

The purpose of this work is to extend the same approach to the approximation of vector data. For simplicity, I will assume that the metric is Euclidean both in the input and in the output space of the approximation, that is:

$$\|v\| = \left( \sum_{i=1}^{N} v_i^2 \right)^{1/2}$$

$$\|x\| = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}$$

The main results can be applied with some precautions to the more general Riemannian metric.

## 3 From basis function to basis fields: Irrotational fields

A natural way to perform a transition from scalar to vector approximation is to begin by assuming that the vectorial data originate from an irrotational field. A vector field, $v(x)$, is said to be irrotational if and only if a scalar potential function, $U(x)$, is defined over $\mathscr{E}$ such that

$$v_i(x) = \partial U / \partial x_i|_x \quad (i = 1, \ldots, N) .$$

An equivalent condition is that

$$\text{curl}(v) = \nabla \wedge v = 0 . \qquad (2)$$

The symbols $\wedge$ and $\nabla$ indicate respectively the external product[4] and the differential operator $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$. Applying this condition to the approximating field, $F(x)$, corresponds to imposing a set of $N(N-1)/2$ differential constraints:

$$\partial F_i(x)/\partial x_j = \partial F_j(x)/\partial x_i \quad i, j = 1, \ldots, N .$$

If these constraints are fulfilled, the approximating field is the gradient of a scalar map:

$$F(x) = \nabla U(x) \qquad (3)$$

Then, one may approximate the potential $U(x)$ by a weighted sum of scalar basis functions, i.e.

$$U(x) = \sum_{i=1}^{I} c_i G(x, t^i) . \qquad (4)$$

Following the approach suggested by Poggio and Girosi (1990a,b), the centers, $t^i$, of the basis functions may be different from the sampling points, $x^i$. However, here we will consider only a fixed set of such centers.

Equation (4) cannot be used for solving the approximation problem since the data are given not as scalar values of $U(x)$ but as vector values for $F(x)$. However, the gradient is a linear operator and one can combine Eqs. (3) and (4) to obtain

$$F(x) = \sum_{i=1}^{K} c_i \phi(x, t^i) \qquad (5)$$

with

$$\phi(x, t^i) = \nabla G(x, t^i) .$$

Unlike Eq. (4), Eq. (5) can be used directly for the field approximation by relating the values of $F$ at the data points, $F(x^j)$ to the vector data $v^j$. Because of the formal analogy of Eq. (5) with the expression of a scalar map as a linear combination of basis functions, I call the gradient fields, $\phi(x, t^i)$, *basis fields*.

The use of basis fields in the representation of the net vector field (5) corresponds to the use of basis functions in the representation of the potential (4). However, one should stress that this correspondence is formally correct only if Eq. (4) is an *exact* representation for the potential (that is, if the potential function falls within the linear span of the basis functions $G(x, t^i)$). Otherwise, the best approximation achieved by Eq. (5) is not necessarily consistent with the best approximation achieved by Eq. (4). In this case the unknown potential function may be represented as

$$U(x) = \sum_{i=1}^{K} c_i G(x, t^i) + \varepsilon(x) \qquad (6)$$

where $\varepsilon(x)$ is an error function which we may assume to be continuous and differentiable. Taking the gradient of

[4] The external product of two $N$-dimensional vectors, $a \wedge b$, is a two-index antisymmetric tensor, $r$, with $r_{i,j} = a_i b_j - a_j b_i$ (here we do not make a distinction between covariant and contravariant indices). In the particular case of $N = 3$, there are only three distinct absolute values, $|r_{i,j}|$. Therefore, in 3D the external product can also be represented as a vector $c$, $c_i = r_{j,k}$ with $i, j, k$ an even permutation of 123

the above expression leads to the corresponding representation of the unknown field, namely

$$F(x) = \sum_{i=1}^{K} c_i \phi_i(x) + \eta(x) \tag{7}$$

with

$$\eta(x) = \nabla \varepsilon(x) .$$

Clearly, the minimization of $\varepsilon(x)$ does not imply the minimization of $\eta(x)$ and vice-versa. It remains an open and interesting question under which conditions on $F(x)$ and $\phi(x, t^i)$, Eq. (5) is a good approximation given that Eq. (4) is. Concerning this question, one may informally observe that the amplitude of the vectorial error, $\eta(x)$, is directly related to the smoothness of $\varepsilon(x)$. A good approximation for $U(x)$ is likely to result in a bad approximation for $F(x)$ if the error function $\varepsilon(x)$ oscillates, as is typically the case with polynomial interpolation. Girosi and Poggio (1990) have pointed out that with a sufficiently large number of Green's functions it is possible to approximate arbitrarily well any continuous multivariate function, $U(x)$, on a compact domain. This implies that as $K \to \infty$, $\varepsilon(x)$ converges uniformly to zero. If the Green's functions correspond to the optimization of smoothness (as is the case with spline interpolation), then it seems reasonable to expect that the error field, $\eta(x)$ will also converge to zero.

A particular instance of basis fields is derived by taking the gradient of radial basis functions (RBFs). A generic RBF has the form:

$$G(x, t^i) = h(\|x - t^i\|) .$$

The corresponding gradient field is

$$\phi(x, t^i) = ((x - t^i)/\|x - t^i\|)(\partial h/\partial \|x - t^i\|) .$$

For any point, $x$, the vector $\phi(x, t^i)$ is directed toward the "center", $t^i$. Thus, the basis fields obtained by taking the gradients of RBFs have central symmetry. Note that the components of $\phi(x, t^i)$ *are not rotationally invariant*, i.e. they are not RBFs.

A set of central basis fields is shown in Fig. 2. Each field is the gradient of a Gaussian potential function:

$$G(x, t^i) = \exp(-(x - t^i)^T (x - t^i)/\sigma^2)$$

$$\phi(x, t^i) = -2(x - t^i) G(x, t^i)/\sigma^2 .$$

In this example, the centers of 16 basis fields, $t^i$, are placed at the intersections of a $4 \times 4$ square grid which covers the domain of the approximation. The variance, $\sigma$, has been determined in relation to the spacing between the centers, $\Delta$, so as to ensure a significant overlap between neighbouring fields:

$$\sigma = \sqrt{2}\Delta . \tag{8}$$

With this variance, the maximum vector amplitude of each basis field is reached at the four nearest surrounding centers. Note that the basis field $\phi(x, t^i)$ vanishes at the center $t^i$. Therefore, this field cannot provide any contribution to the approximation of a vector placed at its center. In contrast, in a scalar approximation a Gaussian basis function provides the most significant contribution precisely at its center.

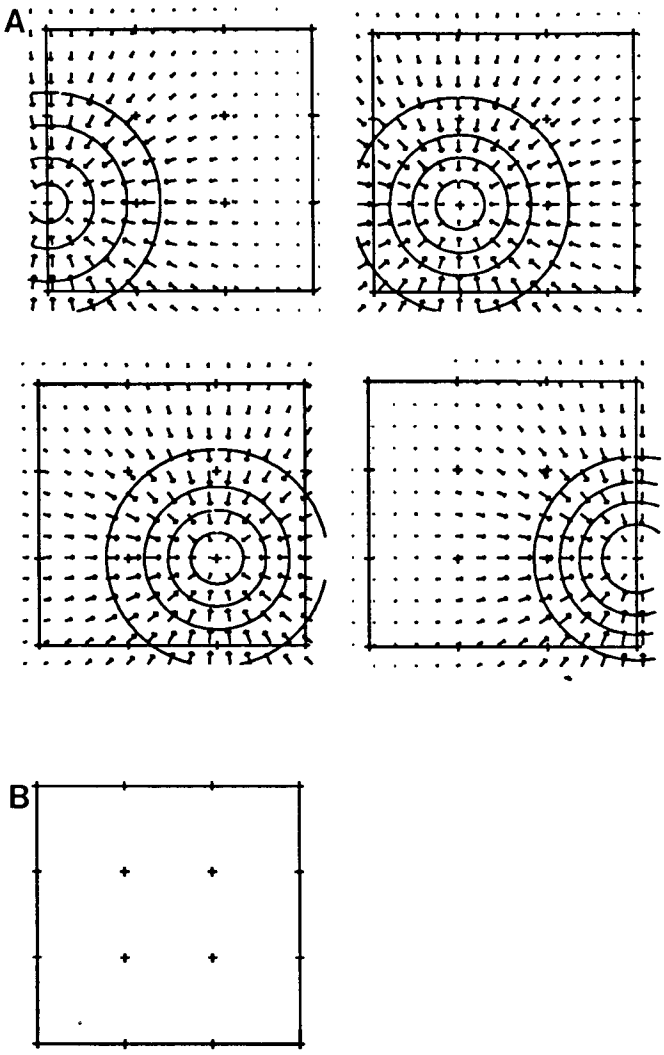


**Fig. 2A, B.** Irrotational basis fields. **A** Four central fields. Each field is the gradient of a Gaussian potential Four isopotential lines surround the center of each Gaussian. **B** Approximation grid. The crosses indicate the center of 16 basis fields covering the input domain

## 4 Approximation algebra

The expression for the approximating field (Eq. 5) is formally equivalent to the expansion of a scalar function into a sum of basis functions. This equivalence becomes more rigorous if the fields, $\phi^i(x)$ are *linearly independent*.

The concept of linear independence for vector fields can be defined operationally as follows. Let us consider a set of $M$ vectors, $v^i$, taken at $M$ locations, $x^i$. Solving the interpolation problem is equivalent to finding a set of parameters, $c_1, c_2, \ldots, c_K$, such that

$$F(x^j) = \sum_{i=1}^{K} c_i \phi(x^j, t^i) \sim v^j$$

for $j = 1, \ldots, M$. This constitutes a set of $M$ vectorial equations which may be expanded into a system of $MN$ scalar equations, one for each vector component. In

vector/matrix notation, this expanded system of equations can be written as

$$\Phi c \sim \hat{v} \tag{9}$$

where we have introduced the (unknown) parameter vector

$$c = (c_1, c_2, \ldots, c_K),$$

the coefficient matrix

$$\Phi = \begin{bmatrix} \phi_1(x^1, t^1) & \phi_1(x^1, t^2) & \ldots & \phi_1(x^1, t^K) \\ \phi_1(x^2, t^1) & \phi_1(x^2, t^2) & \ldots & \phi_1(x^2, t^K) \\ & \ldots & & \\ \phi_1(x^M, t^1) & \phi_1(x^M, t^2) & \ldots & \phi_1(x^M, t^K) \\ & \ldots & & \\ & \ldots & & \\ \phi_N(x^1, t^1) & \phi_N(x^1, t^2) & \ldots & \phi_N(x^1, t^K) \\ \phi_N(x^2, t^1) & \phi_N(x^2, t^2) & \ldots & \phi_N(x^2, t^K) \\ & \ldots & & \\ \phi_N(x^M, t^1) & \phi_N(x^M, t^2) & \ldots & \phi_N(x^M, t^K) \end{bmatrix},$$

and the "data vector

$$\hat{v} = (v_1^1, v_1^2, \ldots, v_1^M, v_2^1, v_2^2, \ldots, v_2^M, \ldots, v_N^1, v_N^2, \ldots, v_N^M). \tag{10}$$

If $K = MN$, the matrix $\Phi$ is square and Eq. (9) can be solved exactly and uniquely for $c$ provided that $\Phi$ be non-singluar. When this condition is fulfilled by any set of distinct points $\{x^i\} \in \mathscr{E}$ we may say that the $K$ fields $\phi^1, \ldots, \phi^K$ are linearly independent and form a basis for a $K$-dimensional functional space. In this case it is strictly correct to characterize the fields $\phi^i(x)$ as *basis fields*. This definition is a direct extension of the concept of basis functions used for scalar multivariable interpolation (Powell 1987; Poggio and Girosi 1990b). An open issue remains to establish under which conditions the matrix $\Phi$ is guaranteed to be non singluar for any choice of sampling points.

Equations (4) and (5) show that the same numerical coefficients appear in the representations of the vector field and of the potential function. Therefore, by using the gradients of basis functions as basis fields, one achieves in a single step the double goal of approximating and integrating[5] a vector field. The integrability conditions, $\partial F_i/\partial x_j = \partial F_j/\partial x_i$ are implicitly satisfied by having chosen a set of basis fields with zero curl.

Figure 3 shows the interpolation of two different sets of data obtained with the basis fields described in Fig. 2. Each data set (Fig. 3, left) consists of 8 randomly generated 2-dimensional vectors. Therefore, there are 16 equations, one per each data component. The interpolation has a unique solutions because there are 16 basis fields and 16 unknown coefficients. The interpolating fields are plotted on the right panels of

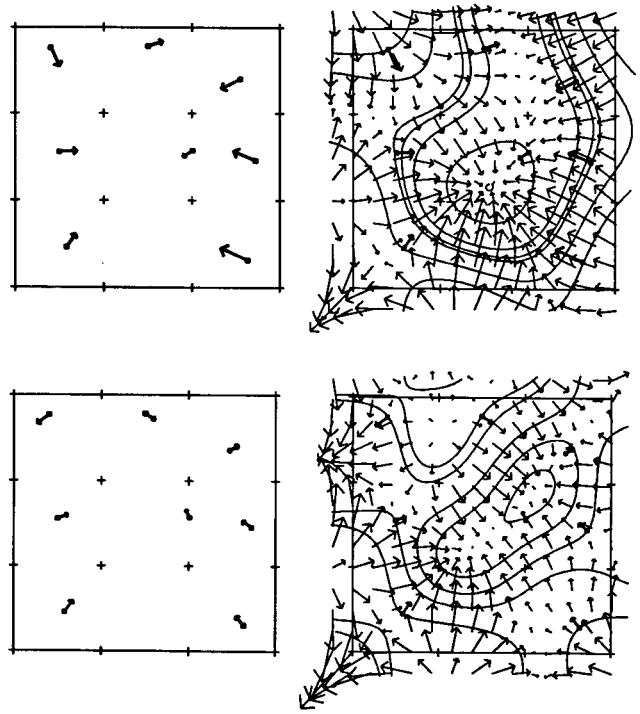

**Fig. 3.** Approximation results. *Left*: Input vector patterns. The two sets of 8 samples (top and bottom) are presented at the same sampling points. *Right*: Approximating fields with isopotential lines. The new fields are combinations of the 16 basis fields described in Fig. 2. These basis fields are sufficient to generate an exact interpolation of eight 2D sampled vectors (there are $8 \times 2 = 16$ equations in 16 unknown coefficients.)

Fig. 3, together with a set of isopotential lines. Each isopotential line corresponds to setting

$$\sum_{i=1}^{K} c_i G(x, t^i) = \text{constant}.$$

These isopotential lines are orthogonal to the data because we are dealing with an exact interpolation.

Note that the matrix $\Phi$ of Eq. (9) depends numerically only upon the locations of the data. The two sets of vector data (top and bottom of Fig. 3) are different. However, the two sets of sampling points, $x^i$, are the same. Therefore, the two interpolating fields have been obtained by using the same inverse matrix, $\Phi^{-1}$. The quality of the interpolation, in terms of the smoothness of the interpolating field, depends critically upon the spatial relation between the locations of the data and the centers of the basis-fields. The high degree of smoothness displayed by these examples is contingent upon the matching between an equally-spaced set of centers and the relatively uniform distribution of the sampling points.

If the number of basis fields is insufficient to interpolate the data – that is, if $K < MN$ – then the Moore-Penrose pseudoinverse[6] of $\Phi$,

$$\Phi^+ = (\Phi^T \Phi)^{-1} \Phi^T \tag{11}$$

[5] A potential function is the integral of its own gradient

[6] If a set of $K$ basis fields is linearly independent with respect to the interpolation problem (i.e. if $K = MN$), then with $MN > K$ the same set of basis field leads to a non-singular approximation. In fact, the Moore-Penrose pseudoinverse can be computed from Eq. (11) since $(\Phi^T \Phi)$ is guaranteed to have full-row rank

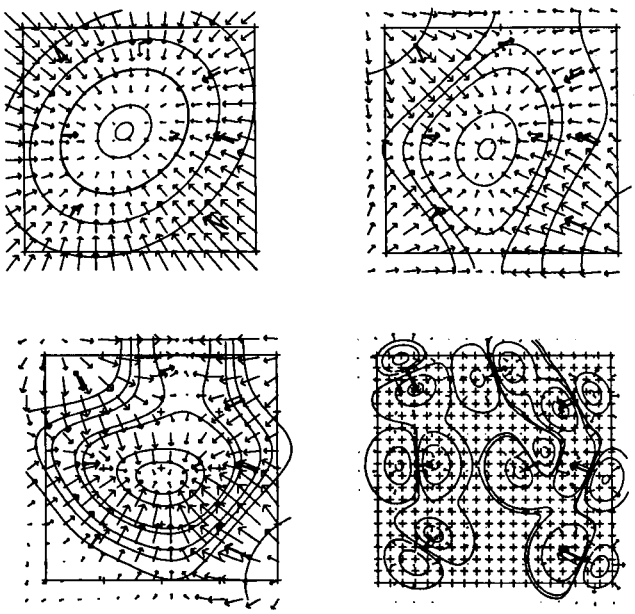

Fig. 4. Approximation results with different sets of basis fields. The same data shown in Fig. 3 (top) are approximated/interpolated by four different sets of basis fields. The centers of the fields are arranged in four different grids over the approximation domain. From top-left to bottom-right: 4 basis fields in a $2 \times 2$ grid, 9 basis fields in a $3 \times 3$ grid, 5 basis fields in a $5 \times 5$ grid, 400 basis fields in a $20 \times 20$ grid. In each grid, the variance of the Gaussian potentials has been established so as to ensure a fixed degree of overlap between neighboring basis fields

generates a least-squares approximation. Using this pseudoinverse corresponds to minimizing the square-error norm

$$\sum_{i=1}^{M} \|F(x^i) - v^i\| . \tag{12}$$

If the number of basis fields is larger than the number of data components (i.e. if $K > MN$), then the interpolation problem admits infinite solutions. In this case the Moore-Penrose pseudoinverse

$$\tilde{\Phi}^+ = \Phi^T(\Phi\Phi^T)^{-1} \tag{13}$$

minimizes the norm of the coefficient vector:

$$\sum_{i=1}^{K} c_i^2$$

Figure 4 illustrates the effect of changing the number of basis fields with respect to the approximation/interpolation of a given vector pattern. The data used in these examples are the same as in Fig. 3 (top): there are 8 vectors at 8 locations, inducing a set of 16 algebraic equations. The approximating domain is covered by one of four different grids of basis-field centers (top-left to bottom right: $2 \times 2$, $3 \times 3$, $5 \times 5$, $20 \times 20$). For each set of Gaussian basis fields, the variance was determined according to Eq. (8), so as to ensure the same degree of overlap across the four examples. The two panels at the top of Fig. 4 show the approximating fields (and potential) obtained by computing the pseudoinverse (11) with 4 and 9 basis fields. In both cases there is a visible approximation error since the degrees of freedom are not sufficient to reproduce exactly the 16 components of the data. As the number of basis fields decreases from 9 to 4, the approximation becomes smoother (as indicated by the isopotential lines) and the data are "corrected" so as to enforce a common central tendency. In contrast, as the set of basis fields becomes increasingly overcomplete (Fig. 4, bottom), the smoothness of the Moore-Penrose interpolation (13) decreases. With 400 basis fields (bottom-right), the interpolation is reminiscent of a look-up table: the field vanishes outside a small region surrounding each data vector. The data are reproduced exactly by attractive/repulsive potential pairs.

## 5 From basis functions to basis fields. Field decomposition

Given a finite number of vectors taken at a set of distinct locations, it is always possible to use the methods described in the previous sections for deriving an irrotational field that interpolates exactly the data. This is a direct consequence of having chosen a set of linearly independent potential functions for generating the basis fields. However, it may well be that the vector field from which the data have been sampled is not irrotational. In this case, what would be the form of a "wrong" irrotational approximation? Let us consider the set of vectors shown at the bottom of in Fig. 5. This set was obtained by taking 20 samples of the field shown in the middle panel. This field is the vector sum of the irrotational field (top-left)

$$F^I(x) = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}(x - \xi) \tag{14}$$

and the circulating field (top-right)

$$F^C(x) = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix}(x - \xi) \tag{15}$$

with $\text{curl}(F^C) = \partial F_2^C/\partial x_1 - \partial F_1^C/\partial x_2 = -2K$.

Figure 6 shows the results of the approximation obtained by combining 36 equally-spaced irrotational basis fields. With 20 vector data, this system of basis fields generates a slightly overcomplete system of 40 equations in 36 unknowns. As expected, the approximating field is remarkably different from the original field. We have achieved the goal of reproducing the data with a good local accuracy. However, the approximating field does not capture at all the smoothness which underlies the sampled pattern.

More generally, it is clearly possible to draw a set of isopotential lines (or even a single isopotential line) which are locally orthogonal to a finite number of vectors. To this end, one needs only to use a sufficiently large number of irrotational basis fields. However, the potential landscape which interpolates a set of circulating vectors is necessarily tortuous and has multiple valleys and peaks. Consequently, the interpolating vector field is doomed to lack the smoothness of the "true"

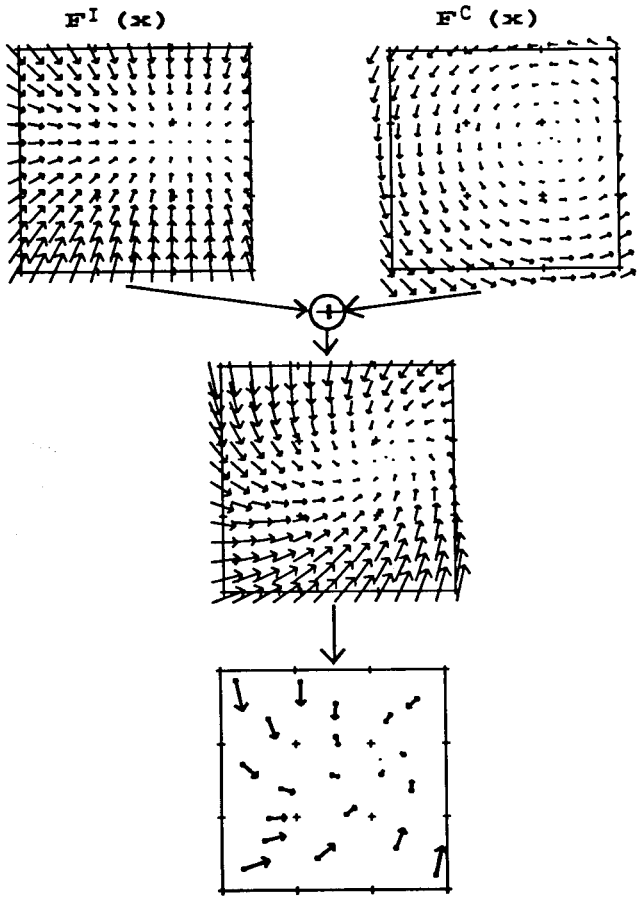


**Fig. 5.** A set of vectorial examples (*bottom*) derived from a non-irrotational continuous field (*middle*). This field is the sum of the two fields shown at the top

**Fig. 6.** Approximation of the samples shown in Fig. 5 (*bottom*) by a combination of 36 irrotational basis fields

rotational field, regardless of the fact that the interpolating potential may optimize a smoothness functional (established by the operator $P$ in Eq. (1)). As the number of basis field is reduced, smoothness may be restored at the expense of a rapidly increasing value of the quadratic error.

Intuitively, in order to achieve an acceptable approximation of a rotational field the computational primitives (i.e. the basis fields) must include among them a number of fields with non-zero curl. In this case one has to give up the idea of reducing the vector approximation to an equivalent approximation of a scalar potential. Thanks to a well-known theorem of potential theory (Kellogg 1953) it is still possible to take advantage of the known properties of scalar basis functions. This theorem states that any continuous vector field, $F(x)$, can be decomposed into the sum of two fields,

$$F(x) = C(x) + S(x) .$$

The first field is irrotational ($\mathrm{curl}(C) = \nabla \wedge C(x) = 0$), whereas the second one is *solenoidal*, that is it has zero divergence ($\mathrm{div}(S) = \nabla \cdot S(x) = 0$). this decomposition is not unique. Two decompositions of $F(x)$ may differ by the gradient of a harmonic function[7].

It is not difficult to prove that in the Euclidean metric a solenoidal field is obtained from an irrotational field when the latter is multiplied by any antisymmetric matrix[8]. This theorem establishes a simple way to generate a set of solenoidal basis fields, $\psi(x, t^i)$, from a set of irrotational ones, $\phi(x, t^i)$. To this end it is sufficient to multiply each irrotational basis field by an antisymmetric matrix, $A$

$$\psi(x, t^i) = A\phi(x, t^i) . \tag{16}$$

From a single set of $K$ irrotational fields it is possible to generate $K_S$ ($\geqslant K$) solenoidal fields by using different antisymmetric matrices (in $2D$, $K_s = K$, since all antisymmetric matrices differ at most by a scaling factor). Then, the representation of a generic continuous field becomes:

$$F(x) = \sum_{i=1}^{K} c_i \phi(x, t^i) + \sum_{i=1}^{K_s} d_i \psi(x, t^i) . \tag{17}$$

Note that the components of the solenoidal fields obtained by the transformation (16) are essentially the same functions of $x$ as the components of the original irrotational fields. Thus, the solenoidal fields form a linearly independent set and the linear algebra described in Sect. 4 can be applied to derive both groups of coefficients, $c_i$ and $d_i$, from a set of vector data.

In two dimensions, an obvious choice for the solenodal basis fields, $\psi(x, t^i)$, is

$$\psi_1(x, t^i) = \phi_2(x, t^i)$$

$$\psi_2(x, t^i) = -\phi_1(x, t^i)$$

Then, $\mathrm{div}(\psi) = \mathrm{curl}_3(\phi) = 0$, and $\mathrm{div}(\phi) = \mathrm{curl}_3(\psi)$.

Figure 7 shows a set of solenoidal fields obtained in this way from the irrotational fields of Fig. 2. With the addition of the solenoidal fields, for each center there

[7] A harmonic function is defined by the Laplace equation, $\Delta\psi(x) = 0$. Since $\Delta\psi = \mathrm{div}(\mathrm{grad}(\psi))$, the gradient of $\psi$ is solenoidal and irrotational at the same time. Therefore, it can be added to $F(x)$ without altering its gradient or its divergence

[8] A matrix, $A = [a_{i,j}]$ is said to be antisymmetric if $a_{ij} = -a_{j,i}$ ($a_{i,i} = 0$). In $2D$, an antisymmetric matrix corresponds to a 90° rotation operator

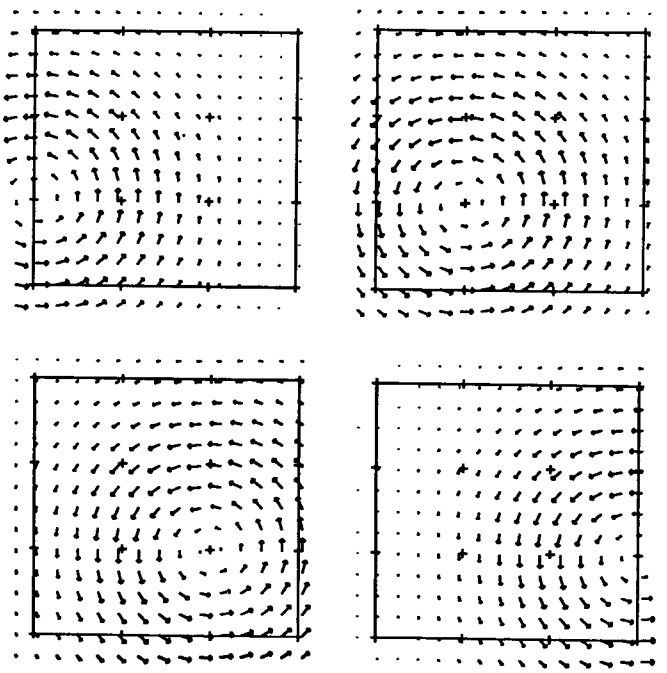

Fig. 7. Four solenoidal basis fields

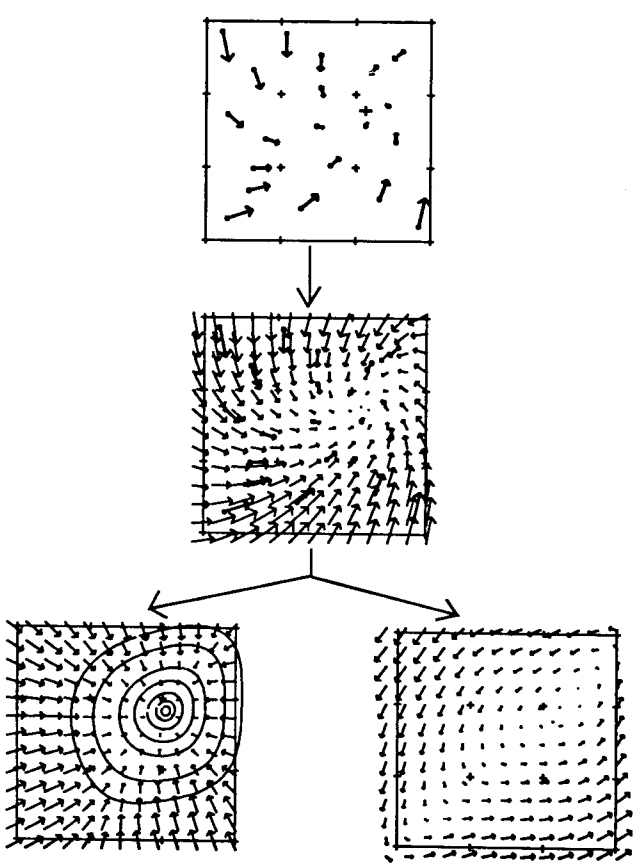

Fig. 8. Approximation results with a combination of irrotational and solenoidal basis fields. *Top*: The pattern of samples. *Middle*: Approximating field. *Bottom left*: Irrotational component with isopotential lines. *Bottom right*: Solenoidal component.

are two basis fields and two coefficients, $c_i$ (irrotational) and $d_i$ (solenoidal.) A set of 32 fields (16 irrotational and 16 solenoidal over a $4 \times 4$ grid) has been used to approximate the same 20 vectors of Fig. 5. The net approximating field is shown in Fig. 8 (center). This field has an obviously simpler (i.e. smoother) structure than the irrotational approximation of the same data (Fig. 6). The combination of solenoidal and irrotational basis fields has clearly captured the main features of the field underlying the pattern of samples.

Once the coefficients have been determined, the two sums of (17) can be plotted separately (Fig. 8, bottom). Thus, the approximation has achieved the three distinct goals: *1)* reproducing the data *2)* decomposing the field into a convergent irrotational component and a rotational component, and *3)* deriving a potential function for the irrotational component. This type of field decomposition may be relevant to computational tasks such as the determination of the center of expansion in an optical flow field. The field underlying the data pattern of Figs. 6 and 8 can indeed be regarded as an optical flow: the observer moves away from a point (the center of the irrotational field (14)), while rotating about the direction of translation. In this case, the irrotational component of the approximating field provides a fairly accurate estimate of the center of expansion (Fig. 8 bottom-left).

## 6 Coordinate transformations

Instead of using basis fields, one could represent each component of the approximating field as a combination of scalar basis functions, $g_1(x), \ldots, g_K(x)$, that is:

$$F_i(x) = \sum_{m=1}^{K} c_{i,m} g_m(x) \qquad i = 1, \ldots, N. \tag{18}$$

In this case, the approximation of a pattern of vectors is reduced to a set of scalar equations of the type

$$v_m^i = \sum_{l=1}^{K} c_{m,l} g_l(x^i) \quad m = 1, \ldots, N \quad i = 1, \ldots, M$$

$$(M = \text{number of data}). \tag{19}$$

However, vector variables such as forces and velocities are not merely collections of numbers. Unlike true scalars[9], they are transformed by a change of coordinates. Consider, for example, a continuous, differentiable and invertible coordinate transformation:

$$\bar{x} = T(x) \tag{20}$$

[9] We remind the reader that a scalar is not simply a single number but a number that remains invariant in a coordinate transformation

characterized locally by the Jacobian matrix, $J(x)$ ($J_{i,j}(x) = \partial T_i/\partial x_j|_x$). Let the data be of the contravariant type[10] (e.g. velocity vectors). In this case each component of the data follows the transformation rule:

$$\bar{v}_j^i = \sum_{m=1}^{N} J_{j,m}(x^i) v_m^i . \tag{21}$$

It is not difficult to see that, in a generic coordinate transformation the representation (18) is bound to be destroyed – the set of coefficients, $c_{m,l}$, must change in order to approximate the same vectors in the new coordinate system. In fact, by combining Eqs. (21) and (19) one obtains the approximation of the vector data in the new coordinate system:

$$\bar{v}_j^i = \sum_{m=1}^{N} J_{j,m}(x^i) \sum_{l=1}^{K} c_{m,l} g_l(x^i) .$$

After carrying out the first summation, the expression for $\bar{v}$ in the new coordinate system becomes:

$$\bar{v}_j^i = \sum_{l=1}^{K} c'_{j,l} g'_l(\bar{x}^i) \tag{22}$$

with

$$c'_{j,l} = \sum_{m=1}^{N} J_{j,m}(x^i) c_{m,l}$$

$$g'_l(\bar{x}^i) = g_l(T^{-1}(\bar{x}^i)) .$$

Therefore, with the representation "by components" (18), both the coefficients and the basis functions must change in order to preserve the approximation after a change of coordinates.

In contrast, let us consider the representation of the approximating field as a combination of basis fields, $\phi^1(x), \phi^2(x), \ldots, \phi^K(x)$:

$$F(x) = \sum_{i=1}^{K} c_i \phi^i(x) . \tag{23}$$

Also, let us assume that these basis fields follow the same transformation law as the data. Then, to approximate contravariant vectors we use contravariant basis fields, whose components transform as:

$$\bar{\phi}_j^l(\bar{x}) = \sum_{m=1}^{N} J_{j,m}(T^{-1}(\bar{x})) \phi_m^l(T^{-1}(\bar{x})) . \tag{24}$$

In the original coordinate system the approximation problem is expressed by a set of equations:

$$v_j^i = \sum_{l=1}^{K} c_l \phi_m^l(x^i) . \tag{25}$$

Unlike Eqs. (19), Eqs. (25) contain a single set of coefficients, $c_i$, for all the field components. In contrast, now each field component has its own set of real-valued basis functions, $\phi_m^l(x)$. By combining Eqs. (21) and (25), one obtains the approximation in the new coordinate system:

$$\bar{v}_j^i = \sum_{m=1}^{N} J_{j,m}(x^i) \sum_{l=1}^{K} c_l \phi_m^l(x^i) .$$

After carrying out the first sum and taking into account the transformation law (24), the above expression becomes

$$\bar{v}_j^i = \sum_{l=1}^{K} c_l \bar{\phi}_j^l(\bar{x}^i) . \tag{26}$$

Note that here the coefficients, $c_i$, have survived the coordinate transformation (20). In the new coordinate system the net vector field is a linear combination of the transformed basis fields with the same set of coefficients. The computational burden associated with the existance of different coordinate systems has been entirely placed in the transformation of the basis fields from one system to another.

A final remark on coordinate transformations regards the smoothness of the approximating field. The basis fields in the new coordinate system may be a distorted version of the original ones. As a consequence, some of their original smoothness properties may be lost in the transformation. Clearly this does not occur with the component based representation (18), since the same basis functions are used in all coordinate systems. At first, this may seem to be a disadvantage of the basis-field approach. However, one should keep in mind that *smoothness is inherently a coordinate-dependent property*. Accordingly, in many instances it seem desirable for the computational primitives used in functional approximation to share the same type of coordinate dependence. For example, Flash and Hogan (1985) have used the coordinate dependence of smoothness as a symmetry principle to infer from the approximation of measured arm trajectories, that multi-joint arm movements are likely to be planned in hand coordinates rather than in joint coordinates.

## 7 Conclusions

The problem of learning an associative mapping from a set of examples has been recently cast into the mathematical framework of multivariate approximation (Poggio and Girosi 1990a). A particular relevance has been attributed to the possibility of expressing the approximating surface as a weighted sum of local basis functions, such as multivariate Gaussians. Each basis function provides a significant contribution only within a limited region, in a way that is consistent with the notion of a receptive field. By combining basis functions, one can reconstruct a scalar function, that is a function with associates each point of its domain to a real number.

[10] According to standard tensor algebra (Levi-Civita 1977), a vector (or, more precisely, a 1-fold system) can either be contravariant (if it transforms according to Eq. (21) or covariant (if it transforms according to the inverse law). For example, in Newtonian mechanics velocity is contravariant and force is covariant. Velocity is contravariant because it transforms in the same way as displacement ($\bar{dx} = J(x)dx$). The covariance of force arises from the fact that power, i.e. the linear form obtained from the inner product of force and velocity, is invariant under coordinate transformations. Therefore, $F = J(x)^T \bar{F}$. The arguments developed in this section apply to covariant vectors without any substantial change

In many important cases of biological relevance, the examples are vectors and the underlying function is a vector-valued map, that is a vector field. A number of experiments have strongly suggested that in order to process vectorial information the central nervous system takes advantage of functional modules expressing specific vector-field patterns. For instance, some investigators (Saito et al. 1986; Tanaka et al. 1989; Andersen et al. 1990) have found neurons in the medial superior temporal area of the visual cortex that responded selectively to specifically structured vector patterns corresponding to expanding/contracting, circulating and translational flows. In a completely different area of research, Bizzi et al. (1991) have suggested that the premotor layers of the frog's spinal cord are organized into distinct functional modules. When activated by microstimulation, each module generates a particular field of forces at a limb's endpoint. A central question shared by researchers in vision and motor control concerns how these elementary vector fields may be combined to analyze or to generate a wider variety of vector patterns.

In this paper I have addressed this question by advocating the use of basis fields as the vectorial counterparts of basis functions. I started by considering the particular case in which the field to be represented is irrotational and can be expressed as the gradient of a scalar potential. In this case, a representation of the potential as a sum of basis functions corresponds to a representation of the fields as a sum of irrotational basis fields obtained as gradients of the same basis functions. I showed that by using a set of irrotational basis fields it is possible to approximate a variety of vector patterns with a different degree of smoothness and accuracy, depending on the number of basis fields relative to the number of examples.

A number of important vector patterns (like the one shown in Fig. 5) are directly related to the presence of a significant curl in the underlying field. However, by using a set of irrotational basis fields one can only generate a total field with zero curl. Therefore, in order to achieve a general representation for continuous fields it is necessary to complement the set of irrotational basis fields with a set of fields with a non-zero curl. Remarkably, this additional set of fields can be obtained by a simple antisymmetric transformation of the irrotational basis fields. Thus, it is possible to reduce the approximation of continuous vector fields to the mathematical tools of scalar-function approximation. A similar approach has been developed by Wahba (1982) who introduced the concept of vector splines on the sphere with the purpose of estimating the vorticity and divergence of the atmosphere from sampled wind-vector data.

Instead of using basis fields, one may be tempted to treat each component of the approximating map as an independent scalar entity to be represented as a combination of local basis functions. In some cases, a disadvantage of this approach is that the values assumed by a vector component are contingent upon the arbitrary choice of a coordinate system. In contrast, most if not all the interesting properties of a vector field do not depend on such a choice. They are invariant under coordinate transformations. For example, a vector field may contain a point attractor or it may be characterized by a circulating pattern. These interesting properties involve *relations* among vector components, such as the relations which define the curl and the divergence of a field. One way to capture the invariant properties expressed by these relations is to use vector fields instead of scalar functions as computational primitives.

*Acknowledgements.* I wish to thank T. Poggio for inspiring this work and for providing many useful comments. I also took advantage of a number of stimulating discussions with many other people. In particular I wish to thank E. Bizzi, E. Fasse, T. Flash, F. Gandolfo, F. Girosi, S. Giszter, N. Hogan, M. Jordan, P. Morasso, R. Shadmehr, T. Sanger and V. Tagliasco. This research was supported by NIH grants NS09343 and AR26710 and by ONR grant N00014/88/K/0372.

## References

Andersen RA, Snowden RJ, Treue S, Graziano M (1990) Hierarchical processing of motion in the visual cortex of monkey. Cold Spring Harbor Symp Quant Biol The Brain 55:741–748

Bizzi E, Mussa-Ivaldi FA, Giszter SF (1991) Computations underlying the execution of movement: A biological perspective. Science 253:287–291

Flash T, Hogan N (1985) The coordination of arm movements: An experimentally confirmed mathematical model. J Neurosci 5:1688–1703

Gibson JJ (1950) The perception of the visual world. Houghton-Mifflin, Boston

Girosi F, Poggio T (1990) Networks and the best approximation property. Biol Cybern 63:169–176

Hogan N (1985) The mechanics of multi-joint posture and movement control. Biol Cybern 52:315–331

Hopfield JJ (1982) Neural networks and physical systems with emergent collective computational abilities. Proc Nat Acad Sci USA 79:2554–2558

Horn BKP (1986) Robot vision. MIT Press, Cambridge

Kellogg OD (1953) Foundations of potential theory. Dover, New York

Levi-Civita T (1977) The absolute differential calculus. (Calculus of tensors). Dover, New York (First English Edition, Blackie and Son, Glasgow 1926)

Micchelli CA (1986) Interpolation of scattered data: Distance matrices and conditionally positive definite functions. Constructive Approx 2:11–22

Mussa-Ivaldi FA, Giszter SF (1992) Vector field approximation: A computational paradigm for motor control and learning. Biol Cybern 67:491–500

Mussa-Ivaldi F, Hogan N, Bizzi E (1985) Neural, mechanical and geometrical factors subserving arm posture in humans. J Neurosci 5:2732–2743

Poggio T (1990) A theory of how the brain might work. Cold Spring Harbor Symp Quant Biol The Brain 55:899–910

Poggio T, Girosi F (1990a) Networks for approximation and learning. Proc. IEEE 78:1481–1497

Poggio T, Girosi F (1990b) A theory of networks for learning. Science 247:978–982

Poggio T, Torre V, Koch C (1985) Computational vision and regularization theory. Nature 317:314–319

Powell MJD (1987) Radial basis functions for multivariable interpolation: a review. In: Mason JC, Cox MG (eds) Algorithms for approximation. Clarendon Press, Oxford

Rice JR (1964) The approximation of functions. Addison-Wesley, Reading

Rumelhard DE, McLelland JL, the PDP Research Group (1986) Parallel distributed processing. Explorations in the microstructure of cognition, MIT Press, Cambridge

Saito H, Yukio M, Tanaka K, Hikosaka K, Fukada Y, Iwai E (1986) Integration of direction signals of image motion in the superior temporal sulcus of the macaque monkey. J Neurosci 6:145–157

Shadmehr R, Muss-Ivaldi FA, Bizzi E (1992) Postural force fields of the human arm and their role in generating multi-joint movements. J Neurosci (in press)

Spivak M (1965) Calculus on manifolds. Benjamin/Cummings Publishing Co., Menlo Park

Tanaka K, Fukada Y, Saito H (1989) Underlying mechanisms of the reponse specificity of the expansion/contraction and rotation cells in the dorsal part of the medial superior temporal area of the macaque monkey. J Neurophysiol 62:642–656

Tikhonov AN, Arsenin VY (1977) Solutions of ill-posed problems. W. H. Winston, Washington

Wahba G (1982) Vector splines on the sphere, with application to the estimation of vorticity and divergence from discrete noisy data. In: Schempp W, Zeller K (eds) Multivariate approximation theory, Birkhauser Verlag, Basel Boston Stuttgart, pp 407–429