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Theoretical and Experimental Perspectives on Arm Trajectory Formation: A Distributed Model of Motor Redundancy

F.A. Mussa-Ivaldi, J. McIntyre and E. Bizzi
Massachusetts Institute of Technology
Dept. of Brain and Cognitive Sciences
Cambridge, MA

Abstract

A model of redundant motor systems is presented in the form of a network of constraint equations expressing the geometrical relations among component elements and their steady-state mechanical behavior. The model takes into account the non-linear features of motor system geometry and is used to represent mechanical interactions with the environment as well as to derive appropriate patterns of control inputs given a wide variety of motor tasks. This approach, which has been derived from the analysis of biological motor systems, can also be applied to the control of artificial manipulators.

1. Introduction

The purpose of the studies described here is to develop a distributed computational framework for representing redundant motor systems (i.e., systems in which there are more independently controlled elements than task dimensions). Redundant motor systems are part of our daily experience: for instance, when we move the hand along a prespecified path or when we exert a force against some object, the number of degrees of freedom and, ultimately, of independently controlled muscles, far exceeds the number of variables (or components) associated with the description of the task.

Several experimental investigations of biological systems (Feldman, 1966, Rack and Westbury, 1969, Bizzi et al., 1976, Nichols and Houk, 1976) have focused on the role of muscle mechanical properties in motor control, suggesting that a muscle is mechanically analogous to a "tunable" spring: i.e., it is characterized by a set of integrable functions between length and tension at steady state.

Neural input selects a particular function (a length/tension curve) out of this set. The equilibrium position and the stiffness of a joint is then defined, for any given value of muscle activation, as the position at which the length-dependent forces of opposing muscles generate equal and opposite torques about the joint. This view of posture has been more recently extended to the analysis of movement and trajectory formation (Bizzi et al, 1984, Hogan, 1985). It has been proposed that arm movements are represented and generated by the central nervous system (CNS) as smooth transitions in posture along "virtual trajectories." The virtual trajectory is given as a time sequence of equilibrium configurations defined by the muscle elastic properties. Experimental and simulation evidence supporting this view has been obtained for single-joint movements (Bizzi et al., 1984) as well as for multi-joint arm trajectories (McKeon et al., 1984, Flash, 1987).

Here, we consider the role of muscle elastic properties from a related but different perspective. We will show that muscle elastic properties lead to a natural representation of motor redundancy. The rationale of our approach is illustrated by a very simple example. Let us consider two springs, having length l_1 and l_2, in a serial arrangement. The total length, L, is the sum of l_1 and l_2. If we ignore the physical properties of this system, the problem of deriving the changes dl_1 and dl_2 given a net change dL is ill-posed, because it has infinite solutions. However, if we consider the elastic behavior of the springs given an externally imposed net change, dL, the problem is no longer ill-posed: the system will settle to a configuration change (dl_1, dl_2) which is at a minimum of potential energy compatible with its geometrical constraints and with the global change. If the elastic elements are also tunable, the new configuration (l_1+dl_1, l_2+dl_2) will be at equilibrium after the control inputs to each spring are set to cancel the induced elastic strains. Through this approach we not only reduce a computationally ill-posed problem to a well-posed one, but we also obtain a description of active coordination which reflects the interactions of the system with the environment.

In this paper we extend this strategy to more complex geometries characterized by non-linear transformations, such as those relating joint angles to hand position or joint angles to muscle lengths. Specifically, we have developed a general description for redundant kinematic structures as networks of elements, each one satisfying a set of local constraint equations. The simulation of these networks allows us to obtain changes in force and position variables within all the elements given a displacement or a force imposed by the environment. We were also able to derive coordinated changes of the control inputs to the elastic elements which are required to implement a trajectory as well as a desired net output force. This goal was achieved by simulating an externally imposed change in position or force of the redundant kinematic structure.

2. The Model

An elastic element can be either a "primitive" element, i.e., an actual system component characterized by controllable spring-like properties, or an aggregate of components which, as a whole, is characterized at steady-state by a dependency of force upon position and/or vice versa.

Primitive elements. A primitive elastic element can be described in two forms: as a stiffness or as a compliance. In the former case, the steady-state force output is a function of position, x, and of the control input u, i.e.:

$$f = f(x, u) \tag{1}$$

It has a local stiffness and compliance given by:

$$k = \frac{\partial f}{\partial x}, \quad c = k^{-1} \text{ for } k \text{ non-zero.} \tag{2}$$

For a compliant element, position is the output variable obtained, always at steady-state, from the values of the applied force and of the control input:

$$x = x(f, u) \tag{3}$$

$$c = \frac{\partial x}{\partial f}, \quad k = c^{-1} \text{ for } c \text{ non-zero.} \tag{4}$$

In our model, we will assume that the processing units representing primitive elastic elements are capable of computing the corresponding functions (1) and (3) for each value taken by the independent variables. Furthermore, each stiffness unit can also perform, at any point (x,u), the following operations: a) compute the sensitivity, $\sigma_f = \frac{\partial f}{\partial u}$; and b) given an externally imposed displacement, dx, derive the change in input which cancels the induced elastic stress ($df = k\,dx$), $du = -\sigma_f^{-1} k\,dx$ (input update).

Similarly, a compliant primitive unit is capable of performing the corresponding operations at any point (f,u), namely: a) compute the sensitivity $\sigma_x = \frac{\partial x}{\partial u}$; and b) given an externally imposed change in force, df, causing a displacement $dx = c\,df$ at constant input, derive the change in input which would achieve the same displacement with no change in the applied force, $du = \sigma_x^{-1} c\,df$ (input update).

It is important to observe that all the operations described above and, in particular, the input updates, are entirely local since each element makes exclusive use

of its own properties to derive the output from the inputs and to calculate updates of the control inputs.

Compound elements. Compound elastic elements, like the primitive elements, are also characterized by force/position relationships. There are, however, three major differences between compound and primitive elements. First, in a compound element, position and force are N-dimensional arrays. Accordingly, stiffness and compliance are NxN matrices. Second, compound elements are not directly accessed by an input signal. Instead, their elastic behavior is established on the basis of their components and of the connection geometry. Third, the force/position relationships are not given explicitly but are derived, locally, on the basis of the force/position relationships of the component elements. In a distributed representation, a compound element is the root node of a tree which exchanges position and force data with lower-order nodes, the direction of these exchanges being determined by the topology of the connections in the corresponding physical system.

Although from the point of view of the environment a compound elastic element can be treated both as a stiffness (position in, force out) and as a compliance (force in, position out), the implementation of the elastic behavior depends upon the topology of the connection with the component elements.

Connection topology and geometrical transformations. Elastic elements can be connected together in any combination of two basic types of topologies. The first, which we refer to as "parallel connection," is defined as a generalized common-position constraint whereas the second, or "serial connection," is defined as a generalized common-force constraint. The equations characterizing one can be obtained from the equations characterizing the other by exchanging the roles of force and position variables. The connection topology defines the elastic behavior of a compound element E from the behavior of a set of component elements $e_i (i = 1, \ldots, N)$. [1]

Parallel connections. The transformations characterizing a parallel connection are:

$$\underline{x}_i = \underline{x}_i(\underline{X});\ i = 1, \ldots, N \tag{5}$$

$$\underline{F} = \sum_{i=1}^{N} \underline{J}_i(\underline{X})^T \underline{f}_i \tag{6}$$

[1] Here, upper and lower case letters are used to distinguish between variables associated to compound elements and variables associated to component elements respectively.

where $\underline{J}_i$ are the jacobians of the transformations (5), i.e., $\underline{J}_i = \frac{D\underline{x}_i}{D\underline{X}}$. Essentially, the first expression represents a generalized common position constraint by indicating that the positions of the component elements are functions of the compound element position. These functions cannot, in general, be inverted to derive $\underline{X}$ from an arbitrary set of $\underline{x}_i$ because there are more constraint equations than unknowns. The second expression is also non-invertible and allows one to derive the compound element force given the forces associated with the component elements. A biological example of parallel connectivity is offered by the relation between muscles and skeleton. Given a set of muscles operating about a number of joints, the joint torques are uniquely derived from the muscle tensions and moment arms. The moment arms are the partial derivatives of the functions mapping joint angles into muscle lengths.

In a parallel connection, the local elastic properties of the components are transformed into the corresponding local properties of the compound element. In particular, the net stiffness $\underline{K}$ about a point $(\underline{X}_0, \underline{F}_0)$ is given by:

$$\underline{K} = \sum_{i=1}^{N} (\underline{\Gamma}_i + \underline{J}_i^T \underline{k}_i \underline{J}_i) \tag{7}$$

The matrices $\underline{\Gamma}_i$ are required to account for non-linearities in the transformations (5) and vanish when these are linear (e.g., with constant moment arms). The physical meaning of the matrix Γ is that of an apparent stiffness introduced in the compound element not by the spring-like properties of the components, but by the non-linear connection geometry. Denoting with $x_{i,k}$ and $f_{i,k}$ the k-th components of the position array, $\underline{x}_i$, and of the force vector, $\underline{f}_i$, associated with the element e_i, the element $\Gamma_{i,j,l}$ of $\underline{\Gamma}_i$ is given by:

$$\Gamma_{i,j,l} = \sum_{k=1}^{Ni} \frac{\partial^2 x_{i,k}}{\partial X_j \partial X_l} f_{i,k} \tag{8}$$

Provided that $\det(\underline{K})$ is non-zero, the compliance of the compound element, which gives the position change resulting from an applied force $\underline{dF}$, is defined as

$$\underline{C} = \underline{K}^{-1}. \tag{9}$$

Serial connections. The description of a serial connection is dual to that of a parallel connection: the position of the compound element is obtained from the positions of the component elements while the forces of the component elements are derived from the compound force through a generalized common force constraint. The equations describing a serial connection are:

$$\underline{X} = \underline{X}(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N) \tag{10}$$

$$\underline{f}_i = \underline{J}_i^T \underline{F} \tag{11}$$

where $\underline{J}_i = \frac{D\underline{X}}{D\underline{x}_i}$. To derive the compound compliance from the compliances of the component elements we must again take into account the non-linearity of Eq. (10) which results in an "apparent stiffness" $\underline{\Gamma}_i$ of the element components:

$$\Gamma_{i,j,l} = \sum_{k=1}^{M} \frac{\partial^2 X_k}{\partial x_{i,j} \partial x_{i,l}} F_k \tag{12}$$

where M is the dimension of the compound element and $j,l = 1 \ldots Ni$ (dimension of the i-th component)[2]. Then, the effective compound compliance is

$$\underline{C} = \sum_{i=1}^{N} \underline{J}_i (\underline{c}_i^{-1} - \underline{\Gamma}_i)^{-1} \underline{J}_i^T \tag{13}$$

where $\underline{c}_i$ is the compliance of the i-th component element. The compound stiffness is derived as

$$\underline{K} = \underline{C}^{-1} \tag{14}$$

wherever $\det(\underline{C})$ is non-zero.

An example of serial connectivity is the transformations between joint coordinates and those of the hand. Since the number of joint coordinates exceeds the number of coordinates needed to specify a position of the hand in space, the transformation from joint angles to hand location cannot be uniquely inverted. Dually, the joint torques are derived from the nef hand force whereas an arbitrary setting of torques may not be consistent with the constraint equations (11).

Significance of the Gamma-matrices. The geometrical transformations for position variables in both serial and parallel cases (Eqs. 10, 5) are not restricted to be linear but only to be sufficiently differentiable (i.e., up to the second partial derivatives). With linear transformations the jacobian matrix is just a matrix of

2 For simplicity, here we assume that, given two distinct elements, $e1$ and $e2$,

$$\frac{\partial^2 X_k}{\partial x_{e1,j} \partial x_{e2,l}} = 0.$$

When this is not the case (as in the transformation from relative joint angles to hand coordinates) we add an intermediate compound element, E, such that $\underline{X}_E = [\underline{x}_{e1}, \underline{x}_{e2}]^T$.

constants, otherwise it is a function of position. As a consequence, when a differential change of force is considered ($\underline{df}_i$ in Eq. 6 or $\underline{dF}$ in Eq. 11), one must take into account that, at steady state, the jacobian will also be changed because the element will move to a new equilibrium position. The Gamma-matrices have been introduced to account for such changes in the jacobian. They have the physical meaning of additional stiffness terms introduced by the non-linearity of the position map.

In a parallel connection, the matrix $\underline{\Gamma}$ corresponds to an additional stiffness "seen" by the environment. To illustrate this, let us consider the case of two ideal force generators pulling on a rope in two opposite directions. These force generators are characterized by zero stiffness and they are supposed to be at equilibrium (i.e., $\underline{f}_1 = \underline{f}_2$). Hence the rope can be displaced in the direction of the forces without encountering any resistance, since the net stiffness along that direction is also zero. However, a displacement orthogonal to the line connecting the two force generators is resisted by a restoring force. Clearly, this force cannot be a consequence of the intrinsic actuator-properties, because the actuators have no stiffness. This elastic force is instead a consequence of the non-linear geometry of the system (given a point outside the straight line connecting the two force generators, the relation between the cartesian coordinates of this point and its distance from any of the two force generators is not linear). A similar interpretation can be given to the matrix $\underline{\Gamma}$ in the serial case.

From the above example it is also evident that the matrix $\underline{\Gamma}$ can play a simplifying role in our computational approach. In fact, without such correction term, the stiffness orthogonal to the line connecting two elastic actuators would be zero and, accordingly, the transverse compliance would be singular. In contrast, the Gamma-matrix, which reflects the physical behavior of the system, provides an additional stiffness term and eliminates this singularity, thus adding stability to the system. Another significant advantage of taking into account the Gamma-matrices is that the differential transformation imposed to (or by) a compound element can be uniquely integrated (see discussion).

Mixed topologies. More complex topologies, expressing both common force and common position constraints, are represented by combining the two basic types of connections in a tree structure where each non-terminal node represents an elastic element connected to its descendants according to one of the basic topologies. An example of such a situation is illustrated in Fig. 1. The geometrical structure (Fig. 1a) is a three joint planar arm with eight muscles: two acting on the shoulder (m1, m2), two on the elbow (m3, m4), two at the wrist (m7, m8) and two acting on both elbow and shoulder joints (m5, m6). The corresponding tree structure is shown in Fig. 1b. The root node represents hand coordinates and the branches are coded according to which variable (position or force) is imposed in that direction (it is implicitly intended that a dual map exists for

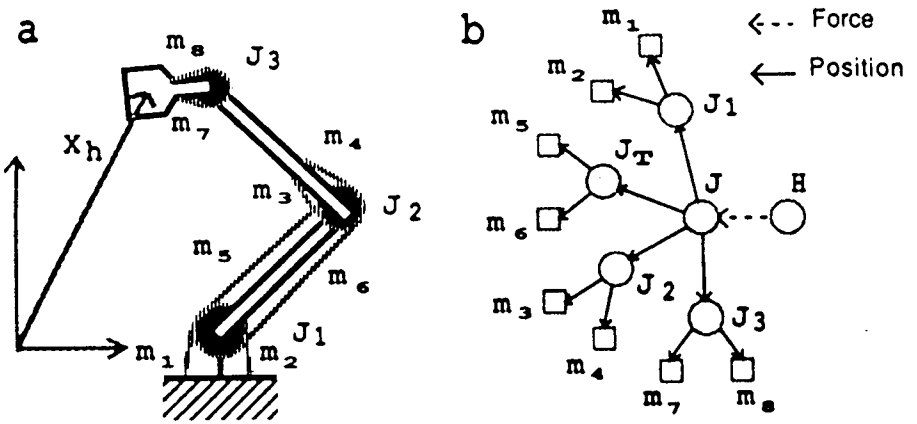


Figure 1. Model arm. a) Physical model. b) Network representation.

information flow in the opposite direction). The hand is connected to a single node representing joint coordinate descriptions (i.e., 3D joint configuration arrays and joint torque vectors). This description is decomposed into four components via projectors (whose duals are injectors applied to torque components to build the net torque vector): a shoulder joint element ($\theta_s = \pi_1(\underline{\theta})$), an elbow joint element ($\theta_e = \pi_2(\underline{\theta})$), a wrist element ($\theta_w = \pi_3(\underline{\theta})$) and a two-joint element ($\underline{\theta}_t = [\pi_1(\underline{\theta}), \pi_2(\underline{\theta})]^T$). Finally, these four nodes are connected to the muscle elements via common position constraints.

3. *Model Behavior*

Interactive behavior and consistency. When a network of elastic elements (all of which are assumed to be in a consistent state) interacts with the environment it behaves, at the interface with the environment, as a single compound elastic element. The environment can either apply a displacement or a force, but not both simultaneously.

An elastic element is said to be in a consistent state when its input and output values correspond to its constraint equations (geometric transformations and elastic properties). An element that is at a consistent state ($\underline{X}^0, \underline{F}^0$) can be driven by the environment to a new consistent state ($\underline{X}^0 + \underline{dX}, \underline{F}^0 + \underline{dF}$) either by an infinitesimal position change, $\underline{dX}$, or by an infinitesimal force change, $\underline{dF}$. The process by which this occurs depends upon whether the element is primarily a stiffness or a compliance. We will consider only the case of a compliant element, as the process for a stiffness element is essentially the same.

Elements which have the properties of a compliance are those in which force is the input and position is the output. These include primitive elements described by equations of the form (3) and compound elements derived from serial connections.

For these elements, a new consistent state for an imposed effort change, $\underline{dF}$, can be computed directly. For the primitive element, $\underline{F}^1 = \underline{F}^0 + \underline{dF}$ and $\underline{X}^1 = \underline{X}(\underline{F}^1)$.

For a compound element operated as a compliance an imposed force change $\underline{dF}$ is directly transformed into the corresponding force changes of the components by using Eq. (11) and taking into account that $\underline{dF}$ causes some change $\underline{dx}_i$ in the component positions. Hence:

$$\underline{df}_i = \underline{J}_i^T \underline{dF} + \underline{\Gamma}_i \underline{dx}_i = (\underline{1} - \underline{\Gamma}_i \underline{c}_i)^{-1} \underline{J}_i^T \underline{dF} \tag{15}$$

where the second term on the right hand side is introduced to account for the change of the jacobian in the displaced position and $\underline{\Gamma}_i$ is defined by the expression (12). Then, the displacement $\underline{dx}_i$ of each component element is:

$$\underline{dx}_i = \underline{c}_i \underline{df}_i = \underline{c}_i \underline{J}_i^T \underline{dF} + \underline{c}_i \underline{\Gamma}_i \underline{dx}_i \tag{16}$$

which, solving for $\underline{dx}_i$ becomes

$$\underline{dx}_i = (\underline{c}_i^{-1} - \underline{\Gamma}_i)^{-1} \underline{J}_i^T \underline{dF} \tag{17}$$

These $\underline{dx}_i$'s correspond to a net displacement of the compound element

$$\underline{dX} = \sum_{i=1}^{N} \underline{J}_i \underline{dx}_i = \sum_{i=1}^{N} \underline{J}_i (\underline{c}_i^{-1} - \underline{\Gamma}_i)^{-1} \underline{J}_i^T \underline{dF} \tag{18}$$

which is identical to the displacement $\underline{C}\underline{dF}$ obtained from the elastic properties of the compound element (Eq. 13).

When a compliance element is operated as a stiffness, a displacement $\underline{dX}$ is imposed by the environment and the corresponding change in force can only be derived as $\underline{dF} = \underline{K}\,\underline{dX}$ (with $\underline{K}$ given by Eq. 4 or Eq. 14). We, therefore, must be in a state where $\underline{K}$ is locally well defined (det($\underline{C}$) is non-zero).

It is apparent from the above discussion that the concept of consistency is a formulation, in terms of constraint satisfaction, of the physical requirement that the potential energy of an elastic element be at a minimum, compatible with the changes imposed by its environment. Similar concepts have been already introduced for describing the behavior of content-addressable memories (Hopfield,

1982) of the Boltzmann machine (Hinton and Sejnowski, 1986) and of other distributed models of cognitive processes (Smolensky, 1986).

Adaptive coordination. For a network of elastic elements, the problem of motor coordination consists in deriving a set of variations, $du_k (k = 1, \ldots, K$ where K is the number of independently controlled primitive elements), of the control inputs to the primitive elements given a desired change, $(\underline{dX}, \underline{dF})$, of the state of the network at the interface with the environment. This is an ill-posed problem because of the excess of control inputs with respect to the dimension of the force and position variables at the environment interface. A way to regularize this problem consists of modifying the inputs according to the local input reset rules of the primitive elements, as the environment imposes the desired global change upon the network.

For example, consider a set of K muscles connected in parallel to a single joint. Let the control inputs to the muscles be at starting values u_k^0 with the joint at a consistent steady state (X^0, F^0) (in this case X is an angle and F is a torque). The goal is to find a set of input changes du_k such that the state becomes $(X^0, F^0 + dF)$ (i.e., a change in force with no change in position as in an isometric force task). Our method solves this problem in three steps. First, the goal is represented by simulating the environment as an ideal force servo acting on the limb at constant input. The network will then settle to a new consistent state, with a position change, as described above. Second, the input update rule is applied to each muscle independently, i.e., $du_k = \sigma_f k_k dx_k$ where the dx_k represent the strains induced by the change of joint angle. Third, with the control inputs at their new values, $u_k + du_k$, all the elements are set to their consistent states while maintaining the force servo applied to the limb. To insure that the correct infinitesimal change is obtained in a single cycle (in this case a joint angle change equal and opposite to that generated in the first step, by the applied force), the force, dF_s, imposed by the external servo may differ, by a linear transformation, from the desired force change, dF to account for nonlinearities in the connection geometry. In this case it is

$$dF_s = \underline{J}^T \underline{k} \underline{J} (\underline{J}^T \underline{k} \underline{J} - \Gamma)^{-1} \underline{dF} \tag{19}$$

where $\underline{k}$ is the muscle stiffness matrix (NxN), $\underline{J}^T$ is the matrix of moment arms $(1 \times N)$ and Γ is the correction term defined by Eq. (8) which here has only one component. If the geometry is linear (i.e., if the muscle moment arms are constant), then $dF_s = dF$. Similar arguments are used to describe the input updates for a desired displacement and for both desired force and displacement in a serial combination of elastic elements.

4. Implementation

A simulator has been constructed for the modelling of distributed elastic networks. The simulator is implemented in Lisp on a Symbolics 3600 computer. The program has been used to simulate the performance of the redundant three joint planar arm described in Section 2 (mixed topologies). A straight line path in cartesian space is computed for the hand. Then, a time series of displacements is imposed on the hand so that it moves along the desired path with a bell shaped velocity profile. At each time step the adaptive coordination algorithm is used to compute an update in muscle inputs so as to keep the system at equilibrium (zero net force). In this case, the input modulates the muscle rest-length through a linear relationship of the form $l_0 = a + bu$, $b < 0$.

The results of this simulation are shown in Fig. 2. The trajectory of the hand is plotted in Fig. 2a, with the configuration of the limb at the endpoints. Figure 2b shows the time series of inputs used to produce the desired movement.

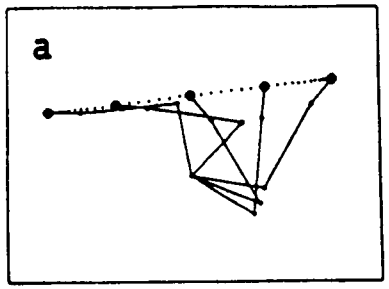
5. Discussion

We have described a model of a redundant motor system characterized by a network of elastic elements. The significance of our model is twofold. First, it can be used for exploring biological motor-control hypotheses since it allows simulation of the observable consequences (that is motions and forces) of neural signals. For instance, the model can be used to explore the roles played in the control of multi-joint posture and movement by different reflex signals such as: (1) feedback depending on muscle length, (2) feedback depending on tendon force, (3) homonimous reflexes which relate the activity in a muscle to the state of the same muscle and (4) heteronimous reflexes which relate the activity of a muscle to the state of a different muscle.

A second significance of our model relates to the planning and control of artificial manipulators. An increasing interest in the area of robotics is being directed to redundant manipulators, that is manipulators possessing more degrees of freedom than that strictly necessary to specify a kinematic task. Redundancy is an anthropomorphic feature which provides more flexibility (e.g., the same endpoint trajectory can be implemented with different trajectories of the joints) at the expense of more complex transformations between task and actuator descriptions. Our model provides a distributed environment to represent and solve these transformations in a way which is consistent with the mechanics of tunable compliant actuators.

a) The arm trajectory for a simulated movement of the 3-joint arm. Tip trajectory is a straight line with a unimodal tangential velocity profile.

b) Pattern of inputs corresponding to the above movement. Inputs specify the muscle rest-lengths. They were computed using the adaptive coordination algorithm.



b

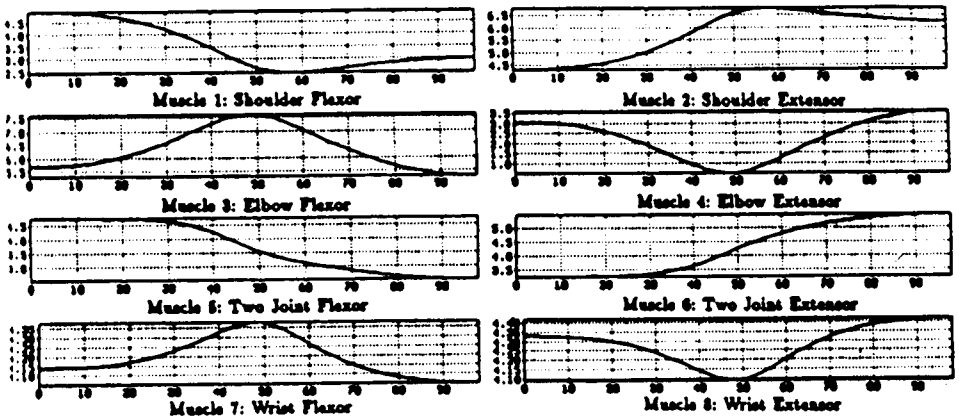


Figure 2. Simulation.

The organization of our network is hierarchical. Each element makes use of information about the structural relations with its components (e.g., the matrix Γ). However, each component is not informed about the structural relations within the element of which itself is a component. From the point of view of a component, the compound element is the "environment" which sets a force (in a serial connection) or a position (in a parallel connection).

The adaptive coordination method can be used to derive patterns of changes in the control inputs from the description of tasks as diverse as applying a desired contact force to an object or generating a sequence of static postures of a limb along a desired trajectory of the end effector.

In the control of kinematically redundant robot arms, the use of pseudoinverses and, in particular, of the More-Penrose pseudoinverse (Ben Israel and Greville, 1980), has been proposed to derive joint motions from the desired movement of the end effector (Whitney, 1969, Klein and Huang, 1983). Recently, a similar

solution has been proposed for biological systems as an application of tensor analysis to the coordinate transformations carried out by the brain (Pellionisz and Llinas, 1985).

Our approach is consistent with the use of weighted pseudoinverses (of which the More-Penrose is a particular case) for generating coordination among the actuators given a desired task in environment coordinates. In a parallel connection, the forces at the elements are derived from the compound force through a chain of transformations which is equivalent to a stiffness-weighted pseudoinverse whereas a compliance-weighted pseudoinverse is obtained when deriving the motion of the elements from that of the compound element in a serial connection. However, in a mixed connection topology (e.g., muscles-joints-hand) the solution contains a combination of both pseudoinverses. Furthermore, our method takes into account explicitly the non-linearity which characterizes certain coordinate transformations by making use of a correction term, the matrix Γ. In the formalism of tensors, this is equivalent to estimating the covariant derivative of the force when the stiffness is transformed between coordinate systems (Brillouin, 1964).

The equivalence of this method with the use of compliance and stiffness weighted pseudoinverses essentially reflects the minimization of the potential energy change associated to an externally imposed displacement. This is the natural solution to redundancy "chosen" by an elastic system interacting with the environment and can be expressed by stating that the metric of such a system is given, at steady state, by its potential energy. Here we propose to adopt the same strategy when generating an active change of the control inputs given a desired task (which is then represented as an externally imposed change).

An advantage of our method with respect to the conventional use of pseudoinverses is that it can provide integrable solutions for the control inputs. Integrability implies that, given an initial condition $q_0 = q(X_0)$, there is a unique map between a position of the end-point and the joint configuration. For example, in the case of a redundant kinematic chain (q = N-dimensional configuration vector, X = M-dimensional end point position vector, $M < N$) it can be shown that the pseudoinverse solution

$$dq = cJ^T(JcJ^T)^{-1}\,dX$$

is integrable if the joint compliance matrix c can be expressed as $c = (k' - \Gamma(q))^{-1}$, where k' is a constant matrix. Different configurations can still be associated to a given end-point location by selecting different elastic properties (i.e., different k and Γ).

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